Strong SOCP Relaxations for the Optimal Power Flow Problem

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FERC Technical Workshop June 24, 2015

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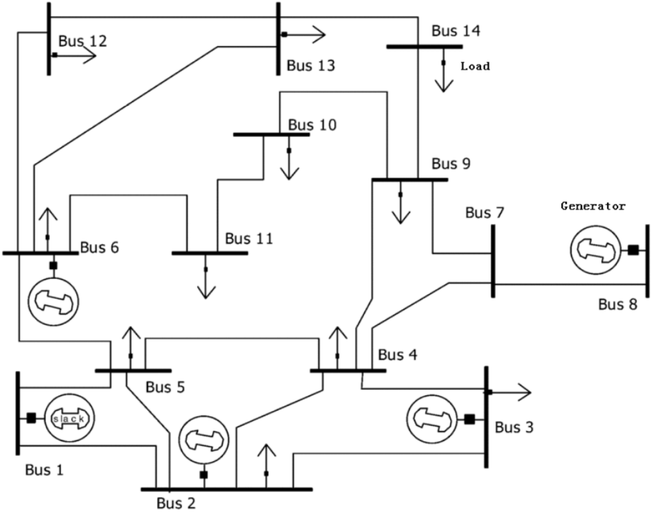
The Optimal Power Flow Problem: Introduction

One-Line Diagram



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Demand: (p_i^d, q_i^d)

Active: $[p_i^{\min}, p_i^{\max}]$

Reactive: $[q_i^{\min}, q_i^{\max}]$

Voltage : $[V_i^{\min}, V_i^{\max}]$

Admittance: (G_{ij}, B_{ij})

Polar formulation

Power flow conservation at each bus:

$$p_i^g - p_i^d = G_{ii} v_i^2 + \sum_{j \in \delta(i)} G_{ij} \left(v_i v_j cos(\theta_i - \theta_j) \right) - \sum_{j \in \delta(i)} B_{ij} \left(v_i v_j sin(\theta_i - \theta_j) \right)$$

$$q_i^g - q_i^d = -B_{ii} v^2 - \sum_{j \in \delta(i)} B_{ij} \left(v_i v_j cos(\theta_i - \theta_j) \right) - \sum_{j \in \delta(i)} G_{ij} \left(v_i v_j sin(\theta_i - \theta_j) \right)$$

Generation and voltage bounds at each bus:

$$\begin{aligned} V_i^{\min} \leq & \; v_i \leq V_i^{\max} \\ p_i^{\min} \leq & \; p_i^g \leq p_i^{\max} \\ q_i^{\min} \leq & \; q_i^g \leq q_i^{\max}. \end{aligned}$$

Objective function:

$$\min \sum_{i \in \mathcal{N}} f_i(p_i^g)$$

Many global solvers do not deal with trigonometric functions.

Rectangular formulation

Power flow conservation at each bus:

$$\begin{aligned} p_i^g - p_i^d &= G_{ii}(e_i^2 + f_i^2) + \sum_{j\in\delta(i)} G_{ij}\,(e_i e_j + f_i f_j) - \sum_{j\in\delta(i)} B_{ij}\,(e_i f_j - f_i e_j) \\ q_i^g - q_i^d &= -B_{ii}(e_i^2 + f_i^2) - \sum_{j\in\delta(i)} B_{ij}\,(e_i e_j + f_i f_j) - \sum_{j\in\delta(i)} G_{ij}\,(e_i f_j - f_i e_j) \end{aligned}$$

Generation and voltage bounds at each bus:

$$\begin{aligned} (V_i^{\min})^2 \le & \quad e_i^2 + f_i^2 & \le (V_i^{\max})^2 \\ p_i^{\min} \le & \quad p_i^g & \le p_i^{\max} \\ q_i^{\min} \le & \quad q_i^g & \le q_i^{\max}. \end{aligned}$$

Objective function:

$$\min \sum_{i\in\mathcal{N}} f_i(p_i^g)$$

Literature

We can categorize the previous work into three classes:

- **Local optimal solutions** based on interior point solvers like MATPOWER.
- **Convex relaxations using semidefinite programming (SDP) and Lasserre relaxations**. (Lavaei and Low, 2012; Madani et. al., 2013; Zhang and Tse, 2012; Lavaei et. al., 2014, Molzahn and Hiskens, 2014).
- **LP and Second order cone program (SOCP) relaxations of polar/rectangular formulations** (Coffrin and Van Hentenryck 2014, Coffrin et. al. 2015)
- **Approximation algorithms with guaranteed bounds for the AC-OPF problem on graphs with bounded tree-width** (Bienstock, Munoz, 2015).
- **Global optimal solutions** based on branch-and-bound (Phan, 2012)

Our Goal

1. Propose strong SOCPs relaxations of OPF which produce dual bounds comparable to the SDPs.
2. Obtain primal OPF feasible solutions with guaranteed performance.

1.1
The Optimal Power Flow Problem: An Alternative formulation

Rectangular formulation: Identifying non-convexities

$$
\begin{aligned}
p_i^g - p_i^d &= G_{ii}(\mathbf{e_i^2} + \mathbf{f_i^2}) + \sum_{j \in \delta(i)} G_{ij} (\mathbf{e_i e_j} + \mathbf{f_i f_j}) - \sum_{j \in \delta(i)} B_{ij} (\mathbf{e_i f_j} - \mathbf{f_i e_j}) \\
q_i^g - q_i^d &= -B_{ii}(\mathbf{e_i^2} + \mathbf{f_i^2}) - \sum_{j \in \delta(i)} B_{ij} (\mathbf{e_i e_j} + \mathbf{f_i f_j}) - \sum_{j \in \delta(i)} G_{ij} (\mathbf{e_i f_j} - \mathbf{f_i e_j}) \\
(V_i^{\min})^2 \leq &\ \mathbf{e_i^2} + \mathbf{f_i^2} \leq (V_i^{\max})^2 \\
p_i^{\min} \leq &\ p_i^g \leq p_i^{\max} \\
q_i^{\min} \leq &\ q_i^g \leq q_i^{\max}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{e_i^2} + \mathbf{f_i^2} &=: \mathbf{c_{ii}} \\
\mathbf{e_i e_j} + \mathbf{f_i f_j} &=: \mathbf{c_{ij}} \\
\mathbf{e_i f_j} - \mathbf{f_i e_j} &=: \mathbf{s_{ij}}.
\end{aligned}
$$

Rectangular formulation: Identifying non-convexities

$$\left.\begin{array}{rcll} p_i^g - p_i^d & = G_{ii}(\mathbf{c_{ii}}) & + \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{s_{ij}}\right) \\ q_i^g - q_i^d & = -B_{ii}(\mathbf{c_{ii}}) & - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{s_{ij}}\right) \\ (V_i^{\min})^2 \le & \mathbf{c_{ii}} & \le (V_i^{\max})^2 \\ p_i^{\min} \le & p_i^g & \le p_i^{\max} \\ q_i^{\min} \le & q_i^g & \le q_i^{\max}. \end{array}\right\} \text{linear}$$

$$\left.\begin{array}{rcl} \mathbf{e_i^2} + \mathbf{f_i^2} & = & \mathbf{c_{ii}} \\ \mathbf{e_i e_j} + \mathbf{f_i f_j} & = & \mathbf{c_{ij}} \\ \mathbf{e_i f_j} - \mathbf{f_i e_j} & = & \mathbf{s_{ij}}. \end{array}\right\} \text{non-convex quadratic.}$$

Implied equations in the c, s space

$$\left.\begin{array}{rcll} p_i^g - p_i^d & = G_{ii}(\mathbf{c_{ii}}) & + \sum_{j\in\delta(i)} G_{ij}\,(\mathbf{c_{ij}}) - \sum_{j\in\delta(i)} B_{ij}\,(\mathbf{s_{ij}}) \\ q_i^g - q_i^d & = -B_{ii}(\mathbf{c_{ii}}) & - \sum_{j\in\delta(i)} B_{ij}\,(\mathbf{c_{ij}}) - \sum_{j\in\delta(i)} G_{ij}\,(\mathbf{s_{ij}}) \\ (V_i^{\min})^2 \le & \mathbf{c_{ii}} & \le (V_i^{\max})^2 \\ p_i^{\min} \le & p_i^g & \le p_i^{\max} \\ q_i^{\min} \le & q_i^g & \le q_i^{\max}. \end{array}\right\} \text{linear}$$

$$\left.\begin{array}{rcl} \mathbf{e_i^2} + \mathbf{f_i^2} & = & \mathbf{c_{ii}} \\ \mathbf{e_i e_j} + \mathbf{f_i f_j} & = & \mathbf{c_{ij}} \\ \mathbf{e_i f_j} - \mathbf{f_i e_j} & = & \mathbf{s_{ij}}. \end{array}\right\} \text{non-convex quadratic.}$$

New implied inequalities:

$$\boxed{\begin{array}{rcl} \mathbf{c_{ij}^2} + \mathbf{s_{ij}^2} & = & \mathbf{c_{ii} c_{jj}} \\ \mathbf{c_{ij}} & = & \mathbf{c_{ji}} \\ \mathbf{s_{ij}} & = & -\mathbf{s_{ji}}. \end{array}}$$

Getting rid of the “e, f" variables

$$
\left.\begin{array}{rcl}
p_i^g - p_i^d & = G_{ii}(\mathbf{c_{ii}}) & + \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{s_{ij}}\right) \\
q_i^g - q_i^d & = -B_{ii}(\mathbf{c_{ii}}) & - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{s_{ij}}\right) \\
(V_i^{\min})^2 \le & \mathbf{c_{ii}} & \le (V_i^{\max})^2 \\
p_i^{\min} \le & p_i^g & \le p_i^{\max} \\
q_i^{\min} \le & q_i^g & \le q_i^{\max}.
\end{array}\right\} \text{linear}
$$

$$
\begin{aligned}
\mathbf{c_{ij}^2} + \mathbf{s_{ij}^2} &= \mathbf{c_{ii}}\mathbf{c_{jj}} \\
\mathbf{c_{ij}} &= \mathbf{c_{ji}} \\
\mathbf{s_{ij}} &= -\mathbf{s_{ji}}.
\end{aligned}
$$

Getting rid of the “e, f" variables

$$
\left.\begin{aligned}
p_i^g - p_i^d &= G_{ii}(\mathbf{c_{ii}}) + \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{s_{ij}}\right) \\
q_i^g - q_i^d &= -B_{ii}(\mathbf{c_{ii}}) - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{s_{ij}}\right) \\
(V_i^{\min})^2 \leq &\quad \mathbf{c_{ii}} \leq (V_i^{\max})^2 \\
p_i^{\min} \leq &\quad p_i^g \leq p_i^{\max} \\
q_i^{\min} \leq &\quad q_i^g \leq q_i^{\max}.
\end{aligned}\right\} \text{linear}
$$

$$
\begin{aligned}
\mathbf{c_{ij}^2} + \mathbf{s_{ij}^2} &= \mathbf{c_{ii}}\mathbf{c_{jj}} \\
\mathbf{c_{ij}} &= \mathbf{c_{ji}} \\
\mathbf{s_{ij}} &= -\mathbf{s_{ji}}.
\end{aligned}
$$

Is this a correct formulation?

Getting rid of the “e, f" variables

$$\left.\begin{aligned}
p_i^g - p_i^d &= G_{ii}(\mathbf{c_{ii}}) + \textstyle\sum_{j\in\delta(i)} G_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{s_{ij}}\right)\\
q_i^g - q_i^d &= -B_{ii}(\mathbf{c_{ii}}) - \textstyle\sum_{j\in\delta(i)} B_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{s_{ij}}\right)\\
(V_i^{\min})^2 \le &\ \mathbf{c_{ii}} \le (V_i^{\max})^2\\
p_i^{\min} \le &\ p_i^g \le p_i^{\max}\\
q_i^{\min} \le &\ q_i^g \le q_i^{\max}.
\end{aligned}\right\} \text{linear}$$

$$\begin{aligned}
\mathbf{c_{ij}^2} + \mathbf{s_{ij}^2} &= \mathbf{c_{ii}}\mathbf{c_{jj}}\\
\mathbf{c_{ij}} &= \mathbf{c_{ji}}\\
\mathbf{s_{ij}} &= -\mathbf{s_{ji}}.
\end{aligned}$$

Above is valid formulation, if $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is tree. In general we also need:

Getting rid of the “e, f" variables

$$\left.\begin{array}{rcll} p_i^g - p_i^d & = G_{ii}(\mathbf{c_{ii}}) & + \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{s_{ij}}\right) \\ q_i^g - q_i^d & = -B_{ii}(\mathbf{c_{ii}}) & - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{s_{ij}}\right) \\ (V_i^{\min})^2 \leq & \mathbf{c_{ii}} & \leq (V_i^{\max})^2 \\ p_i^{\min} \leq & p_i^g & \leq p_i^{\max} \\ q_i^{\min} \leq & q_i^g & \leq q_i^{\max}. \end{array}\right\} \text{linear}$$

$$\begin{aligned} \mathbf{c_{ij}^2} + \mathbf{s_{ij}^2} &= \mathbf{c_{ii}}\mathbf{c_{jj}} \\ \mathbf{c_{ij}} &= \mathbf{c_{ji}} \\ \mathbf{s_{ij}} &= -\mathbf{s_{ji}}. \end{aligned}$$

Above is valid formulation, if $\mathcal{G}(\mathcal{V},\mathcal{E})$ is tree. In general we also need:

$$\left.\sum_{(i,j)\in\text{cycle}} \text{atan2}\left(\frac{\mathbf{s_{ij}}}{\sqrt{\mathbf{c_{ii}}\mathbf{c_{jj}}}}, \frac{\mathbf{c_{ij}}}{\sqrt{\mathbf{c_{ii}}\mathbf{c_{jj}}}}\right) = 0 \;\right\} \text{For all cycles.}$$

Getting rid of the “e, f" variables

$$
\left.\begin{array}{rclll}
p_i^g - p_i^d & = G_{ii}(\mathbf{c_{ii}}) & + \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{s_{ij}}\right) \\
q_i^g - q_i^d & = -B_{ii}(\mathbf{c_{ii}}) & - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{s_{ij}}\right) \\
(V_i^{\text{min}})^2 \leq & \mathbf{c_{ii}} & \leq (V_i^{\text{max}})^2 \\
p_i^{\text{min}} \leq & p_i^g & \leq p_i^{\text{max}} \\
q_i^{\text{min}} \leq & q_i^g & \leq q_i^{\text{max}}.
\end{array}\right\} \text{linear}
$$

$$
\begin{aligned}
\mathbf{c_{ij}^2} + \mathbf{s_{ij}^2} &= \mathbf{c_{ii}}\mathbf{c_{jj}} \\
\mathbf{c_{ij}} &= \mathbf{c_{ji}} \\
\mathbf{s_{ij}} &= -\mathbf{s_{ij}}.
\end{aligned}
$$

$$
\left.\sum_{(i,j)\in\text{cycle}} \text{atan2}\left(\frac{\mathbf{s_{ij}}}{\sqrt{\mathbf{c_{ii}}\mathbf{c_{jj}}}}, \frac{\mathbf{c_{ij}}}{\sqrt{\mathbf{c_{ii}}\mathbf{c_{jj}}}}\right) = 0 \ \right\} \text{For all cycles in a cycle basis.}
$$

This formulation is described in Jabr(2006)

2

Which formulation to use: comparing quality of relaxations

The different formulations

1. Polar formulation [Variables: p, q, v, θ]
2. Rectangular formulation [Variables: p, q, e, f]
3. Alternative formulation [Variables: p, q, c, s]

The different formulations

2. Rectangular formulation [Variables: p, q, e, f]
3. Alternative formulation [Variables: p, q, c, s]

In the following, we relax the "atan2" constraints in Alternative formulation.
Both the above become non-convex quadratic programs.

Standard reformulation

$$
\begin{aligned}
\min \quad & x^T C x + c^T x \\
\text{s.t.} \quad & x^T A_k x + a_k^T x \le b_k \ \forall k \in \{1, \dots, m\} \\
& l \le x \le u.
\end{aligned}
$$

$$
\Leftrightarrow \quad
\begin{aligned}
\min \quad & \langle C, X \rangle + c^T x \\
\text{s.t.} \quad & \langle A_k, X \rangle + a_k^T x \le b_k \ \forall k \in \{1, \dots, m\} \\
& X = x x^T \\
& l \le x \le u.
\end{aligned}
$$

McCormick relaxation: A Linear programming relaxation

$$\begin{aligned} \min \quad & \langle C, X\rangle + c^T x \\ \text{s.t.} \quad & \langle A_k, X\rangle + a_k^T x \leq b_k \ \forall k \in \{1, \dots, m\} \\ & X = xx^T \\ & l \leq x \leq u. \end{aligned}$$

$$\begin{aligned} \min \quad & \langle C, X\rangle + c^T x \\ \text{s.t.} \quad & \langle A_k, X\rangle + a_k^T x \leq b_k \ \forall k \in \{1, \dots, m\} \\ & X_{ij} - l_i x_j - l_j x_i + l_i l_j \geq 0 \\ & u_j x_i - X_{ij} - l_i u_j + l_i x_j \geq 0 \\ & u_i x_j - u_i l_j - X_{ij} + l_j x_i \geq 0 \\ & u_i x_j - u_i x_j - u_j x_i + X_{ij} \geq 0 \\ & l \leq x \leq u. \end{aligned}$$

Denote the McCormick relaxation of the rectangular formulation as $\mathcal{R}_M$.
Denote the McCormick relaxation of the alternative formulation as $\mathcal{A}_M$.

Standard semi-definite programming relaxation

$$
\begin{aligned}
\min \quad & \langle C, X\rangle + c^T x \\
\text{s.t.} \quad & \langle A_k, X\rangle + a_k^T x \le b_k \ \forall k \in \{1, \dots, m\} \\
& X = xx^T \\
& l \le x \le u.
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \langle C, X\rangle + c^T x \\
\text{s.t.} \quad & \langle A_k, X\rangle + a_k^T x \le b_k \ \forall k \in \{1, \dots, m\} \\
& \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0 \\
& l \le x \le u.
\end{aligned}
$$

Denote this SDP relaxation of the rectangular formulation as $\mathcal{R}_{SDP}$.
Denote this SDP relaxation of the alternative formulation as $\mathcal{A}_{SDP}$.
Note: $\mathcal{R}_{SDP}$ is different from $\mathcal{A}_{SDP}$!

Standard second order conic relaxation

$$
\begin{aligned}
\min \quad & \langle C, X\rangle + c^T x \\
\text{s.t.} \quad & \langle A_k, X\rangle + a_k^T x \leq b_k \ \forall k \in \{1, \dots, m\} \\
& X = xx^T \\
& l \leq x \leq u.
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \langle C, X\rangle + c^T x \\
\text{s.t.} \quad & \langle A_k, X\rangle + a_k^T x \leq b_k \ \forall k \in \{1, \dots, m\} \\
& \text{All } 2 \times 2 \text{ principal submatrices of } \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0 \\
& l \leq x \leq u.
\end{aligned}
$$

Denote this SOCP relaxation of the rectangular formulation as $\mathcal{R}_{SOCP}$.
Denote this SOCP relaxation of the alternative formulation as $\mathcal{A}_{SOCP}$.

Classic SOCP relaxation of Alternative formulation

$$\left.\begin{array}{rcll}
p_i^g - p_i^d & = G_{ii}(\mathbf{c_{ii}}) & + \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{s_{ij}}\right) \\
q_i^g - q_i^d & = -B_{ii}(\mathbf{c_{ii}}) & - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{s_{ij}}\right) \\
(V_i^{\min})^2 \leq & \mathbf{c_{ii}} & \leq (V_i^{\max})^2 \\
p_i^{\min} \leq & p_i^g & \leq p_i^{\max} \\
q_i^{\min} \leq & q_i^g & \leq q_i^{\max}.
\end{array}\right\} \text{linear}$$

$$\begin{array}{rcll}
\mathbf{c_{ij}^2} + \mathbf{s_{ij}^2} & = & \mathbf{c_{ii}}\mathbf{c_{jj}} & < - - - - \text{relax this constraint} \\
\mathbf{c_{ij}} & = & \mathbf{c_{ji}} & \\
\mathbf{s_{ij}} & = & -\mathbf{s_{ji}}. &
\end{array}$$

Classic SOCP relaxation of Alternative formulation

$$\left.\begin{array}{rcl} p_i^g - p_i^d & = G_{ii}(\mathbf{c_{ii}}) & + \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{s_{ij}}\right) \\ q_i^g - q_i^d & = -B_{ii}(\mathbf{c_{ii}}) & - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{s_{ij}}\right) \\ (V_i^{\min})^2 \leq & \mathbf{c_{ii}} & \leq (V_i^{\max})^2 \\ p_i^{\min} \leq & p_i^g & \leq p_i^{\max} \\ q_i^{\min} \leq & q_i^g & \leq q_i^{\max}. \end{array}\right\} \text{linear}$$

$$\begin{array}{rcl} \mathbf{c_{ij}^2} + \mathbf{s_{ij}^2} & \leq & \mathbf{c_{ii}}\mathbf{c_{jj}} \\ \mathbf{c_{ij}} & = & \mathbf{c_{ji}} \\ \mathbf{s_{ij}} & = & -\mathbf{s_{ij}}. \end{array}$$

Denote this classic SOCP relaxation of the alternative formulation as $\mathcal{A}^*_{SOCP}$.
Note: $\mathcal{A}^*_{SOCP}$ is different from $\mathcal{A}_{SOCP}$!

Comparison of various relaxations

Theorem

Let $\mathcal{R}_M$, $\mathcal{A}_M$, $\mathcal{R}_{SDP}$, $\mathcal{R}_{SOCP}$, $\mathcal{A}^*_{SOCP}$, $\mathcal{A}_{SDP}$, $\mathcal{A}_{SOCP}$ be the McCormick relaxation of the rectangular formulation, the McCormick relaxation of the alternative formulation, the SDP relaxation of the rectangular formulation, the SOCP relaxation of the rectangular formulation, the classic SOCP relaxation of the alternative formulation, the SDP relaxation of the alternative formulation, and the SOCP relaxation of the alternative formulation respectively. Then:

$$\begin{array}{ccccccccc} \mathcal{R}_{SDP} & \subseteq & \mathcal{R}_{SOCP} & = & \mathcal{A}^*_{SOCP} & \subseteq & \mathcal{A}_{SDP} & \subseteq & \mathcal{A}_{SOCP} \\ & & & & \cap\!| & & & & \\ & & \mathcal{R}^M & \supseteq & \mathcal{A}^M & & & & \end{array}$$

Moreover, there exist instances where each inclusion is proper.

Our choice of relaxation

1. We would like to avoid using SDP relaxations, and prefer SOCP and LP relaxations.
2. It's remarkable that different SDP relaxations have different strength: $\mathcal{R}_{SDP} \subseteq \mathcal{A}^*_{SOCP} \subseteq \mathcal{A}_{SDP}$. Inclusion is strict.
3. Different McCormick-based LP relaxation has different strength: $\mathcal{A}_M \subseteq \mathcal{R}_M$. In our recent work, $\mathcal{A}_M$ is exploited to solve AC-OPF globally.
4. Our choice is $\mathcal{A}^*_{SOCP}$: It provides a natural way to strengthen SOCP relaxations by incorporating convex outer approximations to the nonconvex arctangent constraints.

3

Solving OPF over meshed networks

3.1
A new cycle-based exact formulation for AC-OPF

Revisiting the alternative formulation

$$\left.\begin{array}{rcl} p_i^g - p_i^d & = G_{ii}(\mathbf{c_{ii}}) & + \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{s_{ij}}\right) \\ q_i^g - q_i^d & = -B_{ii}(\mathbf{c_{ii}}) & - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{s_{ij}}\right) \\ (V_i^{\min})^2 \leq & \mathbf{c_{ii}} & \leq (V_i^{\max})^2 \\ p_i^{\min} \leq & p_i^g & \leq p_i^{\max} \\ q_i^{\min} \leq & q_i^g & \leq q_i^{\max}. \end{array}\right\} \text{linear}$$

$$\begin{array}{rcl} \mathbf{c_{ij}^2} + \mathbf{s_{ij}^2} & = & \mathbf{c_{ii}}\mathbf{c_{jj}} \\ \mathbf{c_{ij}} & = & \mathbf{c_{ji}} \\ \mathbf{s_{ij}} & = & -\mathbf{s_{ij}}. \end{array}$$

$$\left.\sum_{(i,j)\in \text{cycle}} \text{atan2}\left(\frac{\mathbf{s_{ij}}}{\sqrt{\mathbf{c_{ii}}\mathbf{c_{jj}}}}, \frac{\mathbf{c_{ij}}}{\sqrt{\mathbf{c_{ii}}\mathbf{c_{jj}}}}\right) = 0 \;\right\} \text{For all cycles in a cycle basis.}$$

The cycle constraint

For a cycle C in the network, instead of satisfying:

$$\sum_{(i,j)\in C} \text{atan2}\left(\frac{\mathbf{s_{ij}}}{\sqrt{\mathbf{c_{ii}}\mathbf{c_{jj}}}}, \frac{\mathbf{c_{ij}}}{\sqrt{\mathbf{c_{ii}}\mathbf{c_{jj}}}}\right) = 0,$$

it is equivalent to guaranteeing “angles sum to zero over the cycle”:

$$\sum_{(i,j)\in C} \theta_{ij} = 2\pi k, \quad \text{for some } k \in \mathbb{Z}. \tag{1}$$

It is sufficient to enforce (1) over cycles in a *cycle basis* (instead of *all* cycles). Condition (1) is equivalent to:

$$\textit{Cycle constraint}: \quad \cos\Big(\sum_{(i,j)\in C} \theta_{ij}\Big) = 1. \tag{2}$$

Cycle constraint (2) can be reformulated as a degree $|C|$ homogeneous *polynomial* $p_C = 0$ in $\mathbf{s_{ij}}$ and $\mathbf{c_{ij}}$ for $(i,j) \in C$.

A new cycle-based exact formulation for OPF

$$\left.\begin{array}{rcll} p_i^g - p_i^d & = G_{ii}(\mathbf{c_{ii}}) & + \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{s_{ij}}\right) \\ q_i^g - q_i^d & = -B_{ii}(\mathbf{c_{ii}}) & - \sum_{j\in\delta(i)} B_{ij}\left(\mathbf{c_{ij}}\right) - \sum_{j\in\delta(i)} G_{ij}\left(\mathbf{s_{ij}}\right) \\ (V_i^{\min})^2 \le & \mathbf{c_{ii}} & \le (V_i^{\max})^2 \\ p_i^{\min} \le & p_i^g & \le p_i^{\max} \\ q_i^{\min} \le & q_i^g & \le q_i^{\max}. \end{array}\right\} \text{linear}$$

$$\begin{aligned} \mathbf{c_{ij}}^2 + \mathbf{s_{ij}}^2 &= \mathbf{c_{ii}}\mathbf{c_{jj}} \\ \mathbf{c_{ij}} &= \mathbf{c_{ji}} \\ \mathbf{s_{ij}} &= -\mathbf{s_{ij}}, \end{aligned}$$

For every cycle C (in a cycle basis): $\mathbf{p_C} = 0$.
This is an exact reformulation of AC-OPF.

Observation

1. New reformulation does not have inverse trigonometric functions, but

   degree of polynomial p_C = size of cycle $|C|$

2. There are standard techniques for converting this into a bilinear program.

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Observation

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2. There are standard techniques for converting this into a bilinear program. This is how BARON solves this program.
3. We have tried a more “natural" way to convert this formulation into a bilinear program: For 3 and 4 cycle also one can show that it is possible to write a bilinear formulation in the space of original variables.
4. For larger cycles, we can decompose it into 3-cycles and 4-cycles.
5. Therefore, all cycle constraints can be written as bilinear constraints.

3- and 4- cycles

1. For a 3-cycle: $\cos(\theta_{12} + \theta_{23} + \theta_{31}) = 1$ can be written as

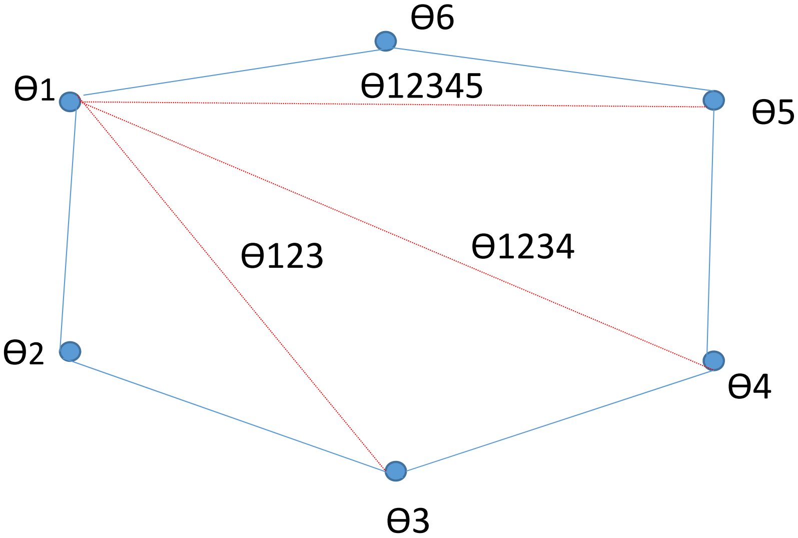
$$\begin{aligned} s_{12}c_{33} + c_{23}s_{31} + s_{23}c_{31} &= 0 \\ c_{12}c_{33} - c_{23}c_{31} + s_{23}s_{31} &= 0. \end{aligned}$$

2. For a 4-cycle: $\cos(\theta_{12} + \theta_{23} + \theta_{34} + \theta_{41}) = 1$ can be written as

$$\begin{aligned} s_{12}c_{34} + c_{12}s_{34} + s_{23}c_{41} + c_{23}s_{41} &= 0 \\ c_{12}c_{34} - s_{12}s_{34} + c_{23}c_{41} - s_{23}s_{41} &= 0. \end{aligned}$$

Larger cycles

Any larger cycle can be decomposed into 3- and 4-cycles:



3.2

Strengthening the $\mathcal{A}^{*}_{SOCP}$ relaxation

Three main ideas

1. McCormick relaxation and LP separation on the new bilinear equations.
2. Arctangent Envelopes: Re-introduce angle variables and devise linear outer-approximation for the following set: (assuming phase differences is in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$)

$$\mathcal{AT} := \left\{(c, s, \theta) \in \mathbb{R}^3 : \theta = \arctan\left(\frac{s}{c}\right), (c, s) \in [\underline{c}, \bar{c}] \times [\underline{s}, \bar{s}]\right\}$$

3. SDP separation: Generate linear constraints to separate SOCP solutions from SDP relaxation.

Idea 1: McComick relaxation and LP separation

Quality of McCormick relaxation and LP separation:

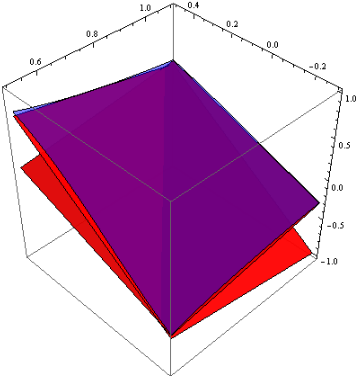
1. We know SDP relaxation is tighter than SOCP relaxation: $\mathcal{R}_{SDP} \subseteq \mathcal{A}^*_{SOCP}$
2. However:
   $\mathcal{R}_{SDP}$ incomparable with $(\mathcal{A}^*_{SOCP} \cap \text{new McCormick constraints})$!
   That is, $\mathcal{R}_{SDP}$ can be worse than $(\mathcal{A}^*_{SOCP} \cap \text{new McCormick constraints})$.

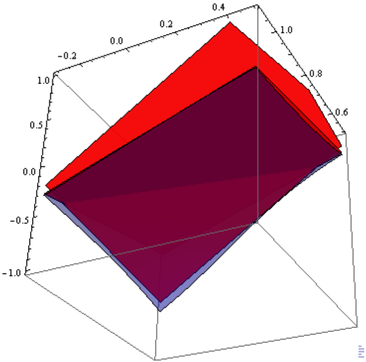
Idea 2: Arctangent Envelopes

1. For each edge (i, j), we want to enforce the arctangent constraint:

$$\mathcal{AT} := \left\{ (c_{ij}, s_{ij}, \theta_{ij}) \in \mathbb{R}^3 : \theta_{ij} = \arctan\left(\frac{s_{ij}}{c_{ij}}\right), (c_{ij}, s_{ij}) \in [\underline{c}_{ij}, \bar{c}_{ij}] \times [\underline{s}_{ij}, \bar{s}_{ij}] \right\}$$

2. Outer approximation of the above set by 4 linear inequalities: Need to solve four simple global optimization problems to obtain these inequalities.





Idea 3: SDP separation

Given a solution (p^*, q^*, c^*, s^*) of $\mathcal{A}^*_{SOCP}$,

1. If there exists a matrix $X^* \succeq 0$, s.t. (c^*, s^*, X^*) satisfies:

$$\begin{aligned} c_{ij} &= X_{ij} + X_{i'j'} & (i,j) \in \mathcal{L} \\ s_{ij} &= X_{ij'} - X_{ji'} & (i,j) \in \mathcal{L} \\ c_{ii} &= X_{ii} + X_{i'i'} & i \in \mathcal{B}, \end{aligned}$$

   where $i' = i + |\mathcal{B}|$ and $j' = j + |\mathcal{B}|$,
   then (c^*, s^*, X^*) is feasible for SDP relaxation $\mathcal{R}_{SDP}$.

2. Otherwise, we can separate $z = (c^*, s^*)$ from the following SDP set $\mathcal{S}$:

$$\mathcal{S} := \left\{ z \in \mathbb{R}^{2|C|} : \exists X \in \mathbb{R}^{2|C| \times 2|C|} \text{ s.t. } -z_l + A_l \bullet X = 0 \quad \forall l \in L,\ X \succeq 0 \right\},$$

   by solving a small SDP over each cycle C in a cycle basis, which produces a linear constraint $\alpha^T z \leq 0$ to be added to $\mathcal{A}^*_{SOCP}$.

4 The business end: computational experiments

Algorithm

1. Input instance in the space of c, s and phase angle θ variables.
2. Compute a cycle basis.
3. For each edge improve bounds on c_{ij}, s_{ij} varaibles by solving SOCPs. **[Parallelized]**
4. For each edge add the arctan linearization to the $\mathcal{A}^*_{AOCP}$ model. **[Parallelized]** [Optional]
5. Solve convex relaxation.
6. For i = 1 to num iterations
    1. Separate cutting planes from each cycle in cycle basis using any one (or both) of the following **[Parallelized]**:
        - McCormick of 3, 4-decomposition of the cycle.
        - SDP separation.
    2. Add cuts and resolve convex relaxation.
7. Use the final (infeasible) solution of convex relaxation as initial point of an interiror point solver (IPOPT) to find a feasible solution to AC-OPF.

Solver, Hardware, Instances

1. Solver: MOSEK, IPOPT.
2. Hardware: 64-bit laptop with Intel Core i7 CPU with 2.00GHz processor and 8 GB RAM.
3. Instances: standard IEEE instances, standard instances where we randomly perturbed the demand $\pm 5\%$, NESTA instances.

Quality of lower bound

Table : Comparison of lower bounds and computation time.

	SDP	SOCP		SOCPA		S34A		SSDP	
case	time	ratio	time	ratio	time	ratio	time	ratio	time
6ww	1.66	0.9937	0.02	0.9998	0.40	0.9999	0.43	1.0000	0.46
9	0.84	1.0000	0.02	1.0000	0.17	1.0000	0.18	1.0000	0.12
9Q	NS	1.0000	0.02	1.0000	0.18	1.0000	0.19	1.0000	0.12
14	1.07	0.9992	0.02	0.9992	0.41	0.9994	0.45	1.0000	0.64
ieee30	1.84	0.9996	0.03	0.9996	0.78	0.9996	0.84	1.0000	1.15
30	2.19	0.9943	0.06	0.9963	0.95	0.9966	1.07	0.9993	1.22
30Q	NS	0.9753	0.07	0.9765	1.02	0.9769	1.11	1.0000	1.32
39	2.20	0.9998	0.04	0.9999	0.90	0.9999	0.99	1.0000	0.72
57	2.60	0.9994	0.04	0.9994	1.43	0.9994	1.47	1.0000	2.14
118	4.58	0.9976	0.11	0.9976	3.69	0.9984	4.83	0.9997	5.19
300	9.81	0.9985	0.21	0.9988	7.62	0.9989	10.40	1.0000	9.83
2383wp	682.86	0.9932	7.11	0.9949	92.83	0.9950	130.03	0.9984	101.31
2736sp	853.92	0.9970	5.14	0.9977	90.93	0.9976	163.80	0.9994	94.48
2737sop	792.25	0.9974	3.85	0.9979	95.28	0.9979	158.80	0.9997	78.70
2746wop	1138.06	0.9963	4.35	0.9971	102.37	0.9973	180.42	0.9995	109.65
2746wp	941.04	0.9967	5.79	0.9975	109.82	0.9975	186.31	0.9998	102.16
3012wp	746.08	0.9936	7.28	0.9946	143.10	0.9946	185.56	0.9974	109.19
3120sp	904.90	0.9955	7.33	0.9962	127.90	0.9965	196.05	0.9987	103.77
3375wp	$>$ 3hr	NA	8.25	NA	149.03	NA	422.35	NA	133.62
Average	**380.37**	**0.9959**	**2.62**	**0.9968**	**48.88**	**0.9970**	**86.59**	**0.9996**	**45.04**

Quality of upper bound

Table : Comparison of upper bounds and percentage optimality gap.

	SDP			SOCP		SOCPA		S34A		SSDP	
case	%gap	time	ratio	%gap	time	%gap	time	%gap	time	%gap	time
6ww	NA	NR	NA	0.63	0.13	0.02	0.48	0.01	0.45	0.00	0.53
9	NA	NR	NA	0.00	0.04	0.00	0.19	0.00	0.20	0.00	0.17
9Q	NA	NR	NA	0.04	0.04	0.04	0.20	0.04	0.21	0.04	0.17
14	0.00	4.49	1.0000	0.08	0.05	0.08	0.44	0.06	0.48	0.00	0.68
ieee30	NA	NR	NA	0.04	0.07	0.04	0.83	0.04	0.88	0.00	1.20
30	0.00	6.54	1.0000	0.57	0.12	0.37	1.01	0.34	1.13	0.07	1.28
30Q	NA	NR	NA	2.48	0.11	2.35	1.07	2.32	1.16	0.00	1.36
39	0.01	5.09	1.0000	0.02	0.10	0.01	0.96	0.01	1.05	0.01	0.78
57	0.00	6.68	1.0000	0.06	0.11	0.06	1.50	0.06	1.55	0.00	2.22
118	0.00	11.16	1.0000	0.25	0.27	0.24	3.86	0.16	5.00	0.03	5.34
300	0.00	22.65	1.0000	0.15	0.62	0.12	8.04	0.11	10.83	0.00	10.33
2383wp	0.68	911.47	0.9969	1.05	21.39	0.89	104.71	0.88	145.29	0.54	124.34
2736sp	0.03	1181.09	0.9997	0.30	16.15	0.23	97.37	0.24	170.92	0.06	114.81
2737sop	0.00	1093.29	1.0000	0.26	12.05	0.21	102.27	0.21	167.59	0.03	103.81
2746wop	0.01	1470.10	0.9999	0.37	9.19	0.29	108.53	0.27	186.91	0.05	138.39
2746wp	0.04	1251.95	0.9996	0.33	14.08	0.25	116.18	0.25	193.91	0.02	124.07
3012wp	0.81	1314.16	0.9934	0.79	19.65	0.70	154.72	0.70	195.56	0.41	134.19
3120sp	0.93	1633.28	0.9916	0.54	16.14	0.47	137.70	0.44	206.20	0.22	121.77
3375wp	NA	$>$3hr	NA	0.26	18.66	0.24	158.21	0.23	431.87	0.13	157.20
Average	**0.19**	**685.53**	**0.9985**	**0.43**	**6.79**	**0.35**	**52.54**	**0.34**	**90.59**	**0.08**	**54.88**

Robustness

Table : Average percentage optimality gaps of perturbed IEEE standard benchmarks.

	SOCP		SOCPA		S34A		SSDP	
case	%gap	time	%gap	time	%gap	time	%gap	time
6ww	0.62	0.06	0.02	0.26	0.01	0.32	0.00	0.46
9	0.00	0.04	0.00	0.19	0.00	0.20	0.00	0.11
9Q	0.09	0.04	0.09	0.19	0.09	0.20	0.09	0.12
14	0.08	0.05	0.08	0.38	0.06	0.41	0.00	0.61
ieee30	0.04	0.06	0.04	0.78	0.04	0.81	0.00	1.03
39	0.03	0.09	0.01	0.91	0.01	0.99	0.00	0.82
57	0.07	0.11	0.07	1.45	0.07	1.51	0.00	1.93
118	0.25	0.30	0.25	3.64	0.17	5.12	0.04	5.04
300	0.63	0.66	0.60	7.90	0.58	13.71	0.33	10.05
2736sp	0.30	12.67	0.23	110.42	0.23	201.81	0.05	120.92
2737sop	0.26	11.98	0.22	108.89	0.22	188.52	0.03	92.18
2746wop	0.38	9.46	0.30	114.89	0.28	215.16	0.06	115.31
2746wp	0.32	12.43	0.25	125.95	0.25	217.88	0.05	117.24
3012wp	0.81	16.77	0.71	125.77	0.71	167.80	0.43	116.88
3120sp	0.53	16.19	0.44	134.98	0.44	180.07	0.25	115.47
3375wp	0.26	19.30	0.24	179.34	0.23	481.92	0.19	161.67
Average	**0.29**	**6.26**	**0.22**	**57.25**	**0.21**	**104.78**	**0.10**	**53.74**

NESTA instances

Table : Comparison of percentage optimality gap for NESTA instances.

	Typical Operating Conditions				Congested Operating Conditions				Small Angle Difference Conditions			
case	SOCP	SOCPA	S34A	SSDP	SOCP	SOCPA	S34A	SSDP	SOCP	SOCPA	S34A	SSDP
3lmbd	1.32	1.25	0.97	0.43	3.30	1.97	1.20	1.31	4.28	2.33	1.51	2.13
4gs	0.00	0.00	0.00	0.01	0.65	0.16	0.12	0.00	4.90	0.42	0.02	0.14
5pjm	14.54	14.47	14.26	6.22	0.45	0.11	0.06	0.00	3.61	0.45	0.34	0.01
6ww	0.63	0.02	0.01	0.00	13.33	0.35	0.14	0.00	0.80	0.02	0.01	0.00
9wscc	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.50	0.43	0.37	0.01
14ieee	0.11	0.11	0.07	0.00	1.35	1.32	1.32	0.00	0.07	0.06	0.06	0.00
29edin	0.14	0.08	0.05	0.00	0.44	0.40	0.36	0.03	34.47	25.94	21.06	31.33
30as	0.06	0.05	0.05	0.00	4.76	2.02	1.89	1.72	9.16	2.43	2.36	0.95
30fsr	0.39	0.23	0.23	0.03	45.97	42.22	41.85	40.28	0.62	0.33	0.27	0.12
30ieee	15.65	5.24	4.79	0.00	0.99	0.86	0.85	0.08	5.87	2.07	1.98	0.00
39epri	0.05	0.02	0.02	0.01	2.99	0.77	0.77	0.00	0.11	0.09	0.09	0.09
57ieee	0.06	0.06	0.06	0.00	0.21	0.21	0.20	0.13	0.11	0.09	0.09	0.05
118ieee	2.10	1.12	0.94	0.25	44.19	40.18	38.22	39.09	12.88	7.77	7.32	9.50
162ieee	4.19	3.99	3.95	3.50	1.52	1.44	1.43	1.20	7.06	5.94	5.81	6.36
189edin	0.22	0.22	0.22	0.07	5.59	3.34	3.33	0.22	2.27	2.21	2.25	1.23
300ieee	1.19	0.78	0.71	0.30	0.85	0.51	0.47	0.15	1.27	0.77	0.70	0.33
1460wp	1.22	1.18	1.18	1.04	1.10	0.98	0.84	0.68	1.37	1.33	1.32	1.22
2224edin	6.22	4.30	4.25	4.60	3.16	2.51	2.43	2.58	6.43	3.91	3.87	4.80
2383wp	1.06	0.87	0.87	0.54	1.12	0.91	0.87	0.52	4.01	2.92	2.80	2.82
2736sp	0.30	0.21	0.20	0.08	1.33	1.14	1.12	0.91	2.34	1.86	1.86	1.92
2737sop	0.26	0.20	0.20	0.03	1.06	0.86	0.86	0.54	2.43	2.23	2.23	1.97
2746wop	0.37	0.28	0.27	0.06	0.49	0.35	0.34	0.17	2.94	2.30	2.31	2.60
2746wp	0.32	0.22	0.22	0.03	0.58	0.34	0.34	0.07	2.44	1.68	1.67	1.83
3012wp	1.04	0.90	0.89	0.50	1.25	0.90	0.89	0.58	2.14	2.00	1.96	1.54
3120sp	0.56	0.45	0.44	0.23	3.03	2.78	2.78	2.34	2.79	2.60	2.57	2.19
3375wp	0.53	0.47	0.46	0.29	0.82	0.64	0.64	0.39	0.53	0.45	0.45	0.28
Average	**5.26**	**3.66**	**3.51**	**1.82**	**8.93**	**6.85**	**6.62**	**6.14**	**5.94**	**3.70**	**3.36**	**4.38**

Some final conclusions

1. Proposed strong SOCP relaxations offer a computationally attractive alternative to SDP relaxation.
2. Lower bounds obtained by strong SOCP relaxations are extremely close to those of SDP relaxation.
3. Solving SOCP based relaxations is orders of magnitude faster than solving SDPs.
4. Given small angle bounds, $\mathcal{A}^*_{SOCP}$ + arctan linearzation + McCormick of cycle constraints is recommended.
5. Without small angle bounds, $\mathcal{A}^*_{SOCP}$ + SDP separation is recommended.
6. The solutions from SOCP relaxation are a good starting point for interior point solver. In comparison, recovering a feasible solution from SDP relaxation is challenging.
7. Proposed strong SOCP relaxations provide stronger bounds than existing quadratic relaxations (e.g. Coffrin et. al. 2015).

Thank You!

For more information, please refer to our papers:

1. B. Kocuk, S. S. Dey, X. A. Sun, Strong SOCP relaxations for the optimal power flow problem, submitted 2015
   https://www.researchgate.net/publication/275588262_Strong_SOCP_Relaxations_for_the_Optimal_Power_Flow_Problem
2. B. Kocuk, S. S. Dey, X. A. Sun, Inexactness of SDP relaxation and valid inequalities for optimal power flow, accepted at *IEEE Trans. Power. Syst.* 2015
   https://www.researchgate.net/publication/276845228_Inexactness_of_SDP_Relaxation_and_Valid_Inequalities_for_Optimal_Power_Flow