A Scalable Semidefinite Relaxation Approach to Grid Scheduling

Ramtin Madani

Department of Electrical Engineering
The University of Texas at Arlington

Alper Atamtürk

Industrial Engineering and Operations Research
University of California, Berkeley

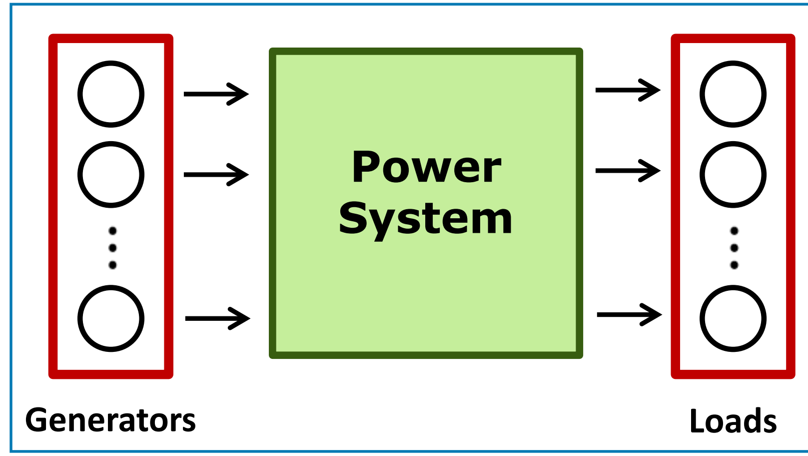
Ali Davoudi

Department of Electrical Engineering
The University of Texas at Arlington

Day-ahead Scheduling

Determine:

1. On/off status of generators,
2. Power injections,
3. Voltages.



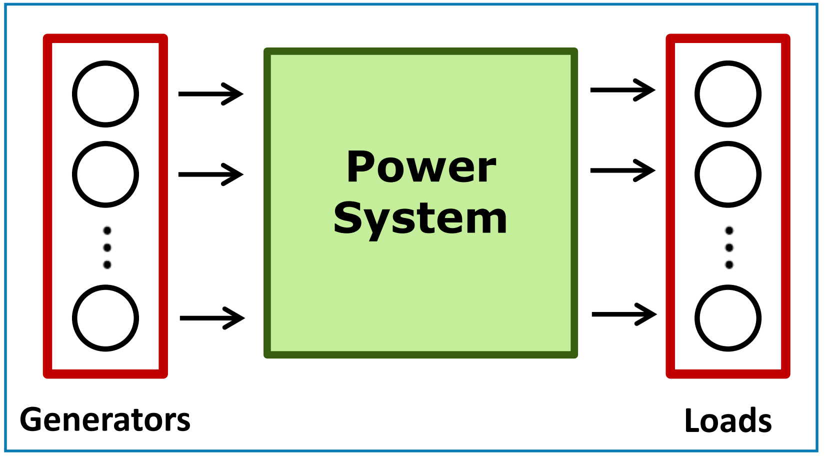
Day-ahead Scheduling

Determine:

1. On/off status of generators,
2. Power injections,
3. Voltages.

Subject to:

1. Unit constraints,
2. Network constraints.



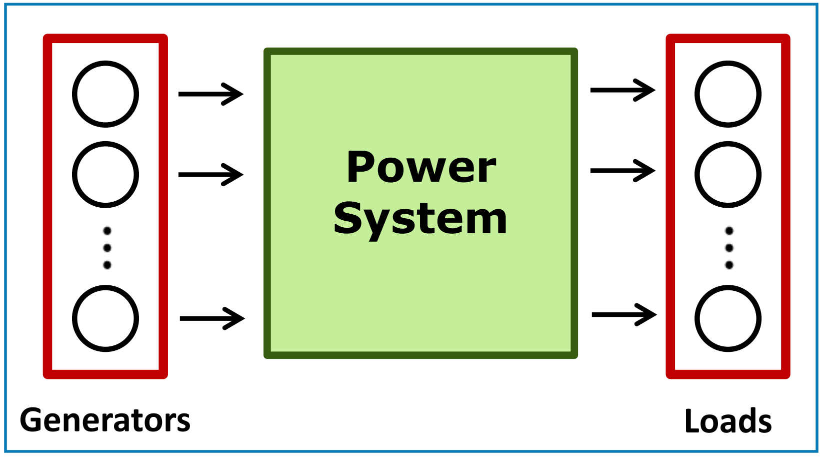
Day-ahead Scheduling

Determine:

1. On/off status of generators,
2. Power injections,
3. Voltages.

Subject to:

1. Unit constraints,
2. Network constraints.



Proposed Approach: Third-order Semidefinite Programming (TSDP).

- In-between the SDP and SOCP relaxations.

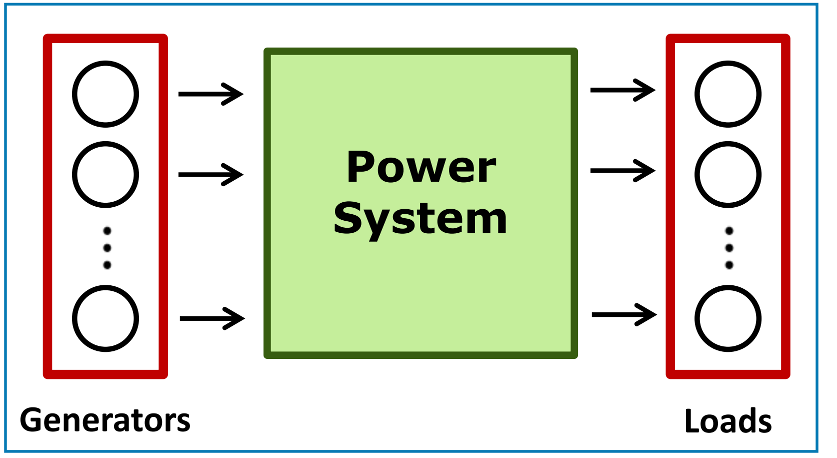
Day-ahead Scheduling

Determine:

1. On/off status of generators,
2. Power injections,
3. Voltages.

Subject to:

1. Unit constraints,
2. Network constraints.



Proposed Approach: Third-order Semidefinite Programming (TSDP).

- In-between the SDP and SOCP relaxations.

Distinctive Features:

- **Compatible with Nonlinearity:** Joint Unit Commitment AC Optimal Power Flow,

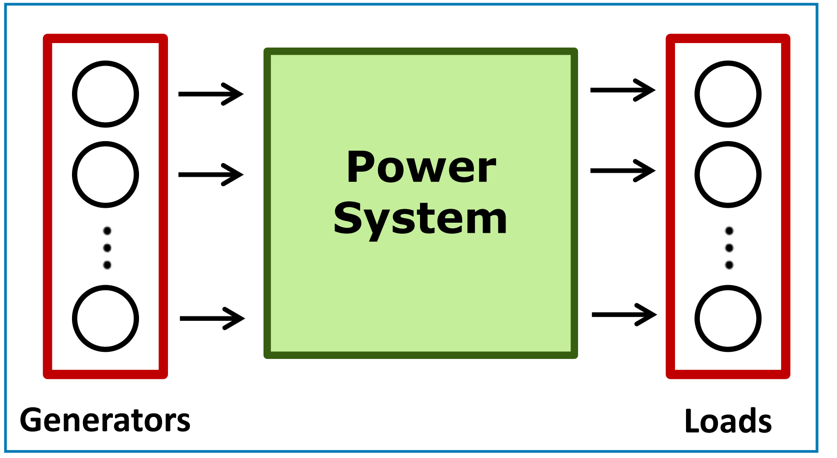
Day-ahead Scheduling

Determine:

1. On/off status of generators,
2. Power injections,
3. Voltages.

Subject to:

1. Unit constraints,
2. Network constraints.



Proposed Approach: Third-order Semidefinite Programming (TSDP).

- In-between the SDP and SOCP relaxations.

Distinctive Features:

- **Compatible with Nonlinearity:** Joint Unit Commitment AC Optimal Power Flow,
- **Massively Scalable:** 24-hour problem with 4000 units in a 13000-bus network,

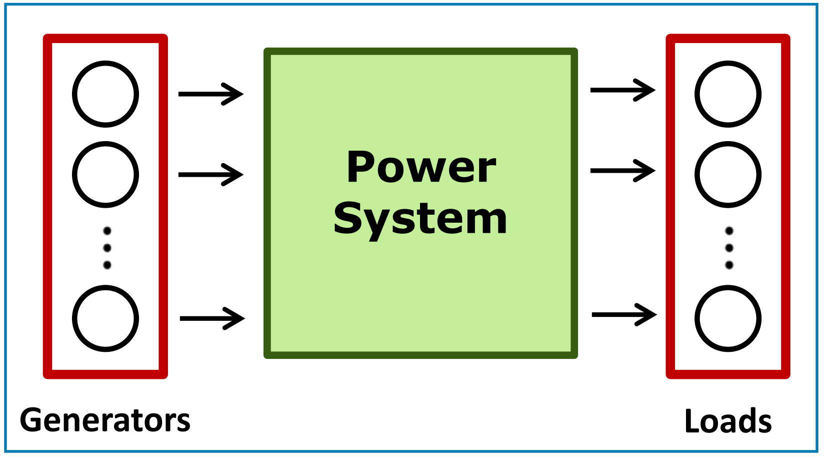
Day-ahead Scheduling

Determine:

1. On/off status of generators,
2. Power injections,
3. Voltages.

Subject to:

1. Unit constraints,
2. Network constraints.



Proposed Approach: Third-order Semidefinite Programming (TSDP).

- In-between the SDP and SOCP relaxations.

Distinctive Features:

- **Compatible with Nonlinearity:** Joint Unit Commitment AC Optimal Power Flow,
- **Massively Scalable:** 24-hour problem with 4000 units in a 13000-bus network,
- **Small Gap:** Less than 2% away from global optimality.

Experimental Results

24-hour problem based on IEEE and European grid data:

		Linear DC Model					Nonlinear AC Model		
Test Case	Number of Units	Ratio of TSDP Inexact Binaries	TSDP Gap	TSDP Time	CPLEX Gap	CPLEX Time	Ratio of TSDP Inexact Binaries	TSDP Gap	TSDP Time
IEEE 118	54	0 / 1296	0%	3s	3.93%	10800s†	0 / 1296	0.01%	11s
IEEE 300	69	0 / 1656	0%	4s	4.08%	10800s†	0 / 1656	0.32%	41s
PEGASE 1354	260	42.3 / 6240	0.06%	18s	8.84%	10800s†	26.5 /6240	1.26%	492s
PEGASE 2869	510	24.5 / 12240	0.09%	35s	17.21%	10800s†	31.1 /12240	0.45%	2199s
PEGASE 9241	1445	31.1 / 34680	0.13%	142s	–	10800s†	68.5 / 34680	1.82%	72024s
PEGASE 13659	4092	71.8 / 98280	0.20%	284s	–	10800s†	91.8 / 98280	1.17%	101450s

† Solver is terminated after 3 hours.

Experimental Results

24-hour problem based on IEEE and European grid data:

		Linear DC Model					Nonlinear AC Model		
Test Case	Number of Units	Ratio of TSDP Inexact Binaries	TSDP Gap	TSDP Time	CPLEX Gap	CPLEX Time	Ratio of TSDP Inexact Binaries	TSDP Gap	TSDP Time
IEEE 118	54	0 / 1296	0%	3s	3.93%	10800s†	0 / 1296	0.01%	11s
IEEE 300	69	0 / 1656	0%	4s	4.08%	10800s†	0 / 1656	0.32%	41s
PEGASE 1354	260	42.3 / 6240	0.06%	18s	8.84%	10800s†	26.5 /6240	1.26%	492s
PEGASE 2869	510	24.5 / 12240	0.09%	35s	17.21%	10800s†	31.1 /12240	0.45%	2199s
PEGASE 9241	1445	31.1 / 34680	0.13%	142s	–	10800s†	68.5 / 34680	1.82%	72024s
PEGASE 13659	4092	71.8 / 98280	0.20%	284s	–	10800s†	91.8 / 98280	1.17%	101450s

† Solver is terminated after 3 hours.

Feasible solutions with less than 2% distance from the globally optimality

Experimental Results

24-hour problem based on IEEE and European grid data:

		Linear DC Model					Nonlinear AC Model		
Test Case	Number of Units	Ratio of TSDP Inexact Binaries	TSDP Gap	TSDP Time	CPLEX Gap	CPLEX Time	Ratio of TSDP Inexact Binaries	TSDP Gap	TSDP Time
IEEE 118	54	0 / 1296	0%	3s	3.93%	10800s†	0 / 1296	0.01%	11s
IEEE 300	69	0 / 1656	0%	4s	4.08%	10800s†	0 / 1656	0.32%	41s
PEGASE 1354	260	42.3 / 6240	0.06%	18s	8.84%	10800s†	26.5 /6240	1.26%	492s
PEGASE 2869	510	24.5 / 12240	0.09%	35s	17.21%	10800s†	31.1 /12240	0.45%	2199s
PEGASE 9241	1445	31.1 / 34680	0.13%	142s	–	10800s†	68.5 / 34680	1.82%	72024s
PEGASE 13659	4092	71.8 / 98280	0.20%	284s	–	10800s†	91.8 / 98280	1.17%	101450s

† Solver is terminated after 3 hours.

Feasible solutions with less than 2% distance from the globally optimality

- Several relaxation schemes have been proposed for UC and OPF:

SDP Relaxation

SOCP Relaxation

Experimental Results

24-hour problem based on IEEE and European grid data:

		Linear DC Model					Nonlinear AC Model		
Test Case	Number of Units	Ratio of TSDP Inexact Binaries	TSDP Gap	TSDP Time	CPLEX Gap	CPLEX Time	Ratio of TSDP Inexact Binaries	TSDP Gap	TSDP Time
IEEE 118	54	0 / 1296	0%	3s	3.93%	10800s†	0 / 1296	0.01%	11s
IEEE 300	69	0 / 1656	0%	4s	4.08%	10800s†	0 / 1656	0.32%	41s
PEGASE 1354	260	42.3 / 6240	0.06%	18s	8.84%	10800s†	26.5 /6240	1.26%	492s
PEGASE 2869	510	24.5 / 12240	0.09%	35s	17.21%	10800s†	31.1 /12240	0.45%	2199s
PEGASE 9241	1445	31.1 / 34680	0.13%	142s	–	10800s†	68.5 / 34680	1.82%	72024s
PEGASE 13659	4092	71.8 / 98280	0.20%	284s	–	10800s†	91.8 / 98280	1.17%	101450s

† Solver is terminated after 3 hours.

Feasible solutions with less than 2% distance from the globally optimality

- Several relaxation schemes have been proposed for UC and OPF:

SDP Relaxation

TSDP Relaxation

SOCP Relaxation

An ideal balance between strength and complexity.

Problem Formulation

- Consider a problem with T time periods and G units:

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\
\text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\
& (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T},
\end{aligned}
$$

Problem Formulation

- Consider a problem with T time periods and G units:

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\
\text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\
& (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T},
\end{aligned}
$$

- The on/off status of unit g at time t : $x_{g,t} \in \{0,1\}$

Problem Formulation

- Consider a problem with T time periods and G units:

$$\begin{aligned} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ \text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g && \forall g \in \mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t && \forall t \in \mathcal{T}, \end{aligned}$$

- The on/off status of unit g at time t : $x_{g,t} \in \{0,1\}$
- Active power injection of unit g at time t : $p_{g,t}$

Problem Formulation

- Consider a problem with T time periods and G units:

$$\begin{aligned} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ \text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T}, \end{aligned}$$

- The on/off status of unit g at time t : $x_{g,t} \in \{0,1\}$
- Active power injection of unit g at time t : $p_{g,t}$
- Reactive power injection of unit g at time t : $q_{g,t}$

Problem Formulation

- Consider a problem with T time periods and G units:

$$\begin{aligned} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ \text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g && \forall g \in \mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t && \forall t \in \mathcal{T}, \end{aligned}$$

- The on/off status of unit g at time t : $x_{g,t} \in \{0,1\}$
- Active power injection of unit g at time t : $p_{g,t}$
- Reactive power injection of unit g at time t : $q_{g,t}$
- Operating cost of unit g at time t : $c_{g,t}$

Problem Formulation

- Consider a problem with T time periods and G units:

$$\begin{aligned} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ \text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T}, \end{aligned}$$

$\Rightarrow$ Overall Cost

- The on/off status of unit g at time t : $x_{g,t} \in \{0,1\}$
- Active power injection of unit g at time t : $p_{g,t}$
- Reactive power injection of unit g at time t : $q_{g,t}$
- Operating cost of unit g at time t : $c_{g,t}$

Problem Formulation

- Consider a problem with T time periods and G units:

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\
\text{subject to} \quad & (\mathbf{x}_{g,:}, \mathbf{p}_{g,:}, \mathbf{q}_{g,:}, \mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\
& (\mathbf{p}_{:,t}, \mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T},
\end{aligned}
$$

→ Unit Constraints

- The on/off status of unit g at time t : $x_{g,t} \in \{0,1\}$
- Active power injection of unit g at time t : $p_{g,t}$
- Reactive power injection of unit g at time t : $q_{g,t}$
- Operating cost of unit g at time t : $c_{g,t}$

Problem Formulation

- Consider a problem with T time periods and G units:

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\
\text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\
& (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T},
\end{aligned}
$$

- The on/off status of unit g at time t : $x_{g,t} \in \{0,1\}$
- Active power injection of unit g at time t : $p_{g,t}$
- Reactive power injection of unit g at time t : $q_{g,t}$
- Operating cost of unit g at time t : $c_{g,t}$
- Unit feasible sets: $\mathcal{U}_1,\mathcal{U}_2,\ldots,\mathcal{U}_G$

Problem Formulation

- Consider a problem with T time periods and G units:

$$\begin{aligned} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ \text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T}, \end{aligned}$$

Network Constraints

- The on/off status of unit g at time t : $x_{g,t} \in \{0,1\}$
- Active power injection of unit g at time t : $p_{g,t}$
- Reactive power injection of unit g at time t : $q_{g,t}$
- Operating cost of unit g at time t : $c_{g,t}$
- Unit feasible sets: $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_G$

Problem Formulation

- Consider a problem with T time periods and G units:

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} & \\
\text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g & \forall g \in \mathcal{G}, \\
& (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t & \forall t \in \mathcal{T},
\end{aligned}
$$

- The on/off status of unit g at time t : $x_{g,t} \in \{0,1\}$
- Active power injection of unit g at time t : $p_{g,t}$
- Reactive power injection of unit g at time t : $q_{g,t}$
- Operating cost of unit g at time t : $c_{g,t}$
- Unit feasible sets: $\mathcal{U}_1,\mathcal{U}_2,\ldots,\mathcal{U}_G$
- Network feasible sets: $\mathcal{N}_1,\mathcal{N}_2,\ldots,\mathcal{N}_T$

Problem Formulation

- Consider a problem with T time periods and G units:

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\
\text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g && \forall g \in \mathcal{G}, \\
& (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t && \forall t \in \mathcal{T},
\end{aligned}
$$

- The on/off status of unit g at time t : $x_{g,t} \in \{0,1\}$
- Active power injection of unit g at time t : $p_{g,t}$
- Reactive power injection of unit g at time t : $q_{g,t}$
- Operating cost of unit g at time t : $c_{g,t}$
- Unit feasible sets: $\mathcal{U}_1,\mathcal{U}_2,\ldots,\mathcal{U}_G$ → Discrete Parameters
- Network feasible sets: $\mathcal{N}_1,\mathcal{N}_2,\ldots,\mathcal{N}_T$

Problem Formulation

- Consider a problem with T time periods and G units:

$$\begin{aligned} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ \text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T}, \end{aligned}$$

- The on/off status of unit g at time t : $x_{g,t} \in \{0,1\}$
- Active power injection of unit g at time t : $p_{g,t}$
- Reactive power injection of unit g at time t : $q_{g,t}$
- Operating cost of unit g at time t : $c_{g,t}$
- Unit feasible sets: $\mathcal{U}_1,\mathcal{U}_2,\ldots,\mathcal{U}_G$ → Discrete Parameters
- Network feasible sets: $\mathcal{N}_1,\mathcal{N}_2,\ldots,\mathcal{N}_T$ → Nonlinear Equations

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1}(1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1} (1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0, 1\}$$

$$\begin{aligned} c_{g,t} = {} & \alpha_g \; p_{g,t}^2 + \beta_g \; p_{g,t} + \\ & \gamma_g \; x_{g,t} + \gamma_g^{\uparrow} \; (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \; x_{g,t-1} (1 - x_{g,t}), \end{aligned}$$

$$\begin{aligned} \underline{p}_g \; x_{g,t} \leq p_{g,t} \leq \bar{p}_g \; x_{g,t} \\ \underline{q}_g \; x_{g,t} \leq q_{g,t} \leq \bar{q}_g \; x_{g,t}. \end{aligned}$$

$$\begin{aligned} x_{g,t} &\geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - \; m_g^{\uparrow} \; + 1, \ldots, t\}, \\ 1 - x_{g,t} &\geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - \; m_g^{\downarrow} \; + 1, \ldots, t\}, \end{aligned}$$

$$\begin{aligned} p_{g,t} - p_{g,t-1} &\leq r_g \; x_{g,t-1} + s_g \; (1 - x_{g,t-1}), \\ p_{g,t-1} - p_{g,t} &\leq r_g \; x_{g,t} + s_g \; (1 - x_{g,t}), \end{aligned}$$

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1} (1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

α_g, β_g : cost coefficients

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1} (1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

α_g, β_g : cost coefficients

$\gamma_g, \gamma_g^{\uparrow}, \gamma_g^{\downarrow}$: fixed/startup/shutdown costs

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1}(1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

α_g, β_g : cost coefficients

$\gamma_g, \gamma_g^{\uparrow}, \gamma_g^{\downarrow}$: fixed/startup/shutdown costs

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1} (1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

α_g, β_g : cost coefficients

$\gamma_g, \gamma_g^{\uparrow}, \gamma_g^{\downarrow}$: fixed/startup/shutdown costs

$\bar{p}_g, \underline{p}_g$: active power upper/lower limits

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1} (1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$

$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$

$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$

$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

α_g, β_g : cost coefficients

$\gamma_g, \gamma_g^{\uparrow}, \gamma_g^{\downarrow}$: fixed/startup/shutdown costs

$\bar{p}_g, \underline{p}_g$: active power upper/lower limits

$\bar{q}_g, \underline{q}_g$: reactive power upper/lower limits

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1}(1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

α_g, β_g : cost coefficients

$\gamma_g, \gamma_g^{\uparrow}, \gamma_g^{\downarrow}$: fixed/startup/shutdown costs

$\bar{p}_g, \underline{p}_g$: active power upper/lower limits

$\bar{q}_g, \underline{q}_g$: reactive power upper/lower limits

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1} (1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \le p_{g,t} \le \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \le q_{g,t} \le \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \ge x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \ge x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \le r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \le r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

α_g, β_g : cost coefficients

$\gamma_g, \gamma_g^{\uparrow}, \gamma_g^{\downarrow}$: fixed/startup/shutdown costs

$\bar{p}_g, \underline{p}_g$: active power upper/lower limits

$\bar{q}_g, \underline{q}_g$: reactive power upper/lower limits

$m_g^{\uparrow}, m^{\downarrow}$: minimum up/down limits

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1} (1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

α_g, β_g : cost coefficients

$\gamma_g, \gamma_g^{\uparrow}, \gamma_g^{\downarrow}$: fixed/startup/shutdown costs

$\bar{p}_g, \underline{p}_g$: active power upper/lower limits

$\bar{q}_g, \underline{q}_g$: reactive power upper/lower limits

$m_g^{\uparrow}, m^{\downarrow}$: minimum up/down limits

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1} (1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

α_g, β_g : cost coefficients

$\gamma_g, \gamma_g^{\uparrow}, \gamma_g^{\downarrow}$: fixed/startup/shutdown costs

$\bar{p}_g, \underline{p}_g$: active power upper/lower limits

$\bar{q}_g, \underline{q}_g$: reactive power upper/lower limits

$m_g^{\uparrow}, m^{\downarrow}$: minimum up/down limits

r_g, s_g : ramp rate limits

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1} (1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

α_g, β_g : cost coefficients

$\gamma_g, \gamma_g^{\uparrow}, \gamma_g^{\downarrow}$: fixed/startup/shutdown costs

$\bar{p}_g, \underline{p}_g$: active power upper/lower limits

$\bar{q}_g, \underline{q}_g$: reactive power upper/lower limits

$m_g^{\uparrow}, m^{\downarrow}$: minimum up/down limits

r_g, s_g : ramp rate limits

Unit Constraints

Unit Constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (1 - x_{g,t-1}) x_{g,t} + \gamma_g^{\downarrow} \, x_{g,t-1}(1 - x_{g,t}),$$

$$\underline{p}_g \, x_{g,t} \leq p_{g,t} \leq \bar{p}_g \, x_{g,t}$$
$$\underline{q}_g \, x_{g,t} \leq q_{g,t} \leq \bar{q}_g \, x_{g,t}.$$

$$x_{g,t} \geq x_{g,\tau} - x_{g,\tau-1}, \quad \forall \tau \in \{t - m_g^{\uparrow} + 1, \ldots, t\},$$
$$1 - x_{g,t} \geq x_{g,\tau-1} - x_{g,\tau}, \quad \forall \tau \in \{t - m_g^{\downarrow} + 1, \ldots, t\},$$

$$p_{g,t} - p_{g,t-1} \leq r_g \, x_{g,t-1} + s_g \, (1 - x_{g,t-1}),$$
$$p_{g,t-1} - p_{g,t} \leq r_g \, x_{g,t} + s_g \, (1 - x_{g,t}),$$

α_g, β_g : cost coefficients

$\gamma_g, \gamma_g^{\uparrow}, \gamma_g^{\downarrow}$: fixed/startup/shutdown costs

$\bar{p}_g, \underline{p}_g$: active power upper/lower limits

$\bar{q}_g, \underline{q}_g$: reactive power upper/lower limits

$m_g^{\uparrow}, m^{\downarrow}$: minimum up/down limits

r_g, s_g : ramp rate limits

Definition. *For every generating unit $g \in \mathcal{G}$, define $\mathcal{U}_g$ to be the set of all quadruplets $(\mathbf{x}_{g,:}, \mathbf{p}_{g,:}, \mathbf{q}_{g,:}, \mathbf{c}_{g,:}) \in \mathbb{R}^{T \times 4}$ that satisfies the unit constraints, for all $t \in \mathcal{T}$.*

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\,\mathbf{v}_t\mathbf{v}_t^*\,\mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t\,\mathbf{v}_t\mathbf{v}_t^*\,\overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$
$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t\,\mathbf{v}_t\mathbf{v}_t^*\,\overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\mathrm{real}\{\mathbf{d}_t\} + \mathbf{B}_t\,\boldsymbol{\theta}_t = \mathbf{C}^\top\mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t\,\boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^* \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \mathbf{v}_t\mathbf{v}_t^* \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$
$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \mathbf{v}_t\mathbf{v}_t^* \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\mathrm{real}\{\mathbf{d}_t\} + \mathbf{B}_t\,\boldsymbol{\theta}_t = \mathbf{C}^\top\mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t\,\boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\, \mathbf{v}_t \mathbf{v}_t^* \; \mathbf{Y}_t^* \} = \mathbf{C}^\top (\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$
$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\mathrm{real}\{\mathbf{d}_t\} + \mathbf{B}_t \, \boldsymbol{\theta}_t = \mathbf{C}^\top \mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t \, \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\, \mathbf{v}_t \mathbf{v}_t^* \; \mathbf{Y}_t^* \} = \mathbf{C}^\top (\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\mathrm{real}\{\mathbf{d}_t\} + \mathbf{B}_t \; \boldsymbol{\theta}_t = \mathbf{C}^\top \mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t \; \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \text{diag}\{\, \mathbf{v}_t \mathbf{v}_t^* \; \mathbf{Y}_t^* \} = \mathbf{C}^\top (\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overrightarrow{\mathbf{Y}}_t^* \}| \leq \mathbf{f}_{\max;t}$$
$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overleftarrow{\mathbf{Y}}_t^* \}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\text{real}\{\mathbf{d}_t\} + \mathbf{B}_t \; \boldsymbol{\theta}_t = \mathbf{C}^\top \mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t \; \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

$\mathbf{d}_t$: nodal power demand

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\, \mathbf{v}_t \mathbf{v}_t^* \; \mathbf{Y}_t^* \} = \mathbf{C}^\top (\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{ \overrightarrow{\mathbf{C}}_t \, \mathbf{v}_t \mathbf{v}_t^* \; \overrightarrow{\mathbf{Y}}_t^* \}| \leq \mathbf{f}_{\max;t}$$
$$|\mathrm{diag}\{ \overleftarrow{\mathbf{C}}_t \, \mathbf{v}_t \mathbf{v}_t^* \; \overleftarrow{\mathbf{Y}}_t^* \}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\mathrm{real}\{\mathbf{d}_t\} + \mathbf{B}_t \, \boldsymbol{\theta}_t = \mathbf{C}^\top \mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t \, \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

$\mathbf{d}_t$: nodal power demand

$\mathbf{C}, \overrightarrow{\mathbf{C}}_t, \overleftarrow{\mathbf{C}}_t$: unit/line incidence matrices

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\, \mathbf{v}_t \mathbf{v}_t^* \; \mathbf{Y}_t^* \} = \mathbf{C}^\top (\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$
$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\mathrm{real}\{\mathbf{d}_t\} + \mathbf{B}_t \, \boldsymbol{\theta}_t = \mathbf{C}^\top \mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t \; \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

$\mathbf{d}_t$: nodal power demand

$\mathbf{C}, \overrightarrow{\mathbf{C}}_t, \overleftarrow{\mathbf{C}}_t$: unit/line incidence matrices

$\mathbf{Y}_t, \overrightarrow{\mathbf{Y}}_t, \overleftarrow{\mathbf{Y}}_t$: nodal/line admittance matrices

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\, \mathbf{v}_t \mathbf{v}_t^* \; \mathbf{Y}_t^* \} = \mathbf{C}^\top (\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\mathrm{real}\{\mathbf{d}_t\} + \mathbf{B}_t \, \boldsymbol{\theta}_t = \mathbf{C}^\top \mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t \; \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

$\mathbf{d}_t$: nodal power demand

$\mathbf{C}, \overrightarrow{\mathbf{C}}_t, \overleftarrow{\mathbf{C}}_t$: unit/line incidence matrices

$\mathbf{Y}_t, \overrightarrow{\mathbf{Y}}_t, \overleftarrow{\mathbf{Y}}_t$: nodal/line admittance matrices

$\mathbf{v}_{\min}, \mathbf{v}_{\max}$: voltage magnitude limits

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\ \mathbf{v}_t\mathbf{v}_t^*\ \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t\ \mathbf{v}_t\mathbf{v}_t^*\ \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$
$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t\ \mathbf{v}_t\mathbf{v}_t^*\ \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\mathrm{real}\{\mathbf{d}_t\} + \mathbf{B}_t\ \boldsymbol{\theta}_t = \mathbf{C}^\top\mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t\ \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

$\mathbf{d}_t$: nodal power demand

$\mathbf{C}, \overrightarrow{\mathbf{C}}_t, \overleftarrow{\mathbf{C}}_t$: unit/line incidence matrices

$\mathbf{Y}_t, \overrightarrow{\mathbf{Y}}_t, \overleftarrow{\mathbf{Y}}_t$: nodal/line admittance matrices

$\mathbf{v}_{\min}, \mathbf{v}_{\max}$: voltage magnitude limits

$\mathbf{f}_{\max;t}$: line thermal limits

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\ \mathbf{v}_t\mathbf{v}_t^*\ \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t\ \mathbf{v}_t\mathbf{v}_t^*\ \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$
$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t\ \mathbf{v}_t\mathbf{v}_t^*\ \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\mathrm{real}\{\mathbf{d}_t\} + \mathbf{B}_t\ \boldsymbol{\theta}_t = \mathbf{C}^\top\mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t\ \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

$\mathbf{d}_t$: nodal power demand

$\mathbf{C}, \overrightarrow{\mathbf{C}}_t, \overleftarrow{\mathbf{C}}_t$: unit/line incidence matrices

$\mathbf{Y}_t, \overrightarrow{\mathbf{Y}}_t, \overleftarrow{\mathbf{Y}}_t$: nodal/line admittance matrices

$\mathbf{v}_{\min}, \mathbf{v}_{\max}$: voltage magnitude limits

$\mathbf{f}_{\max;t}$: line thermal limits

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \text{diag}\{\ \mathbf{v}_t\mathbf{v}_t^*\ \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t\ \mathbf{v}_t\mathbf{v}_t^*\ \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$
$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t\ \mathbf{v}_t\mathbf{v}_t^*\ \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\text{real}\{\mathbf{d}_t\} + \mathbf{B}_t\ \boldsymbol{\theta}_t = \mathbf{C}^\top\mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t\ \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

$\mathbf{d}_t$: nodal power demand

$\mathbf{C}, \overrightarrow{\mathbf{C}}_t, \overleftarrow{\mathbf{C}}_t$: unit/line incidence matrices

$\mathbf{Y}_t, \overrightarrow{\mathbf{Y}}_t, \overleftarrow{\mathbf{Y}}_t$: nodal/line admittance matrices

$\mathbf{v}_{\min}, \mathbf{v}_{\max}$: voltage magnitude limits

$\mathbf{f}_{\max;t}$: line thermal limits

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \text{diag}\{\ \mathbf{v}_t\mathbf{v}_t^*\ \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t\ \mathbf{v}_t\mathbf{v}_t^*\ \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t\ \mathbf{v}_t\mathbf{v}_t^*\ \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\text{real}\{\mathbf{d}_t\} + \mathbf{B}_t\ \boldsymbol{\theta}_t = \mathbf{C}^\top\mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t\ \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

$\mathbf{d}_t$: nodal power demand

$\mathbf{C}, \overrightarrow{\mathbf{C}}_t, \overleftarrow{\mathbf{C}}_t$: unit/line incidence matrices

$\mathbf{Y}_t, \overrightarrow{\mathbf{Y}}_t, \overleftarrow{\mathbf{Y}}_t$: nodal/line admittance matrices

$\mathbf{v}_{\min}, \mathbf{v}_{\max}$: voltage magnitude limits

$\mathbf{f}_{\max;t}$: line thermal limits

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\ \mathbf{v}_t\mathbf{v}_t^*\ \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t\ \mathbf{v}_t\mathbf{v}_t^*\ \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$
$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t\ \mathbf{v}_t\mathbf{v}_t^*\ \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\mathrm{real}\{\mathbf{d}_t\} + \mathbf{B}_t\ \boldsymbol{\theta}_t = \mathbf{C}^\top\mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t\ \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

$\mathbf{d}_t$: nodal power demand

$\mathbf{C}, \overrightarrow{\mathbf{C}}_t, \overleftarrow{\mathbf{C}}_t$: unit/line incidence matrices

$\mathbf{Y}_t, \overrightarrow{\mathbf{Y}}_t, \overleftarrow{\mathbf{Y}}_t$: nodal/line admittance matrices

$\mathbf{v}_{\min}, \mathbf{v}_{\max}$: voltage magnitude limits

$\mathbf{f}_{\max;t}$: line thermal limits

Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \text{diag}\{\, \mathbf{v}_t \mathbf{v}_t^* \; \mathbf{Y}_t^* \} = \mathbf{C}^\top (\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$
$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t \; \mathbf{v}_t \mathbf{v}_t^* \; \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min} \leq |\mathbf{v}_t| \leq \mathbf{v}_{\max}$$

DC Network Constraints:

$$\mathbf{q}_t = 0$$

$$\text{real}\{\mathbf{d}_t\} + \mathbf{B}_t \, \boldsymbol{\theta}_t = \mathbf{C}^\top \mathbf{p}_t$$

$$|\vec{\mathbf{B}}_t \; \boldsymbol{\theta}_t| \leq \mathbf{f}_{\max;t}$$

$\mathbf{d}_t$: nodal power demand

$\mathbf{C}, \overrightarrow{\mathbf{C}}_t, \overleftarrow{\mathbf{C}}_t$: unit/line incidence matrices

$\mathbf{Y}_t, \overrightarrow{\mathbf{Y}}_t, \overleftarrow{\mathbf{Y}}_t$: nodal/line admittance matrices

$\mathbf{v}_{\min}, \mathbf{v}_{\max}$: voltage magnitude limits

$\mathbf{f}_{\max;t}$: line thermal limits

Definition. *For every time interval $t \in \mathcal{T}$, define $\mathcal{N}_t$ to be the set of all pairs $(\mathbf{p}_{:,t}, \mathbf{q}_{:,t}) \in \mathbb{R}^{T\times 2}$, for which there exists a vector of complex voltages $\mathbf{v}_{:,t} \in \mathbb{C}^{\mathcal{V}}$ satisfying the network constraints.*

Convex Relaxation

- Consider a problem with T time periods and G units:

$$\begin{aligned} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ \text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T}, \end{aligned}$$

Convex Relaxation

- Consider a problem with T time periods and G units:

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\
\text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\
& (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T},
\end{aligned}
$$

Convex Relaxation

- Consider a problem with T time periods and G units:

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\
\text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\
& (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T},
\end{aligned}
$$

Convex Relaxation

- Consider a problem with T time periods and G units:

$$\begin{aligned} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ \text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g && \forall g \in \mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t && \forall t \in \mathcal{T}, \end{aligned}$$

- A convex relaxation can be created by replacing $\mathcal{U}_g$ and $\mathcal{N}_t$ with their convex surrogates.

Convex Relaxation

- Consider a problem with T time periods and G units:

$$
\begin{array}{lll}
\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} & \displaystyle\sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} & \\
\text{subject to} & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g & \forall g \in \mathcal{G}, \\
 & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t & \forall t \in \mathcal{T},
\end{array}
$$

- A convex relaxation can be created by replacing $\mathcal{U}_g$ and $\mathcal{N}_t$ with their convex surrogates.
- Unit feasible sets: $\mathcal{U}_g$ → $\mathcal{U}_g^{\text{TSDP}}$

Convex Relaxation

- Consider a problem with T time periods and G units:

$$\begin{array}{lll} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} & \\ \text{subject to} & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:})\in\mathcal{U}_g & \forall g\in\mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t})\in\mathcal{N}_t & \forall t\in\mathcal{T}, \end{array}$$

- A convex relaxation can be created by replacing $\mathcal{U}_g$ and $\mathcal{N}_t$ with their convex surrogates.
- Unit feasible sets: $\mathcal{U}_g$ → $\mathcal{U}_g^{\text{TSDP}}$
- Network feasible sets: $\mathcal{N}_t$ → $\mathcal{N}_t^{\text{SDP}}$ $\mathcal{N}_t^{\text{SOCP}}$

Convex Relaxation

- Consider a problem with T time periods and G units:

$$\begin{array}{lll} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} & \displaystyle\sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} & \\ \text{subject to} & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g & \forall g \in \mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t & \forall t \in \mathcal{T}, \end{array}$$

- A convex relaxation can be created by replacing $\mathcal{U}_g$ and $\mathcal{N}_t$ with their convex surrogates.
- Unit feasible sets: $\mathcal{U}_g$ ➡ $\mathcal{U}_g^{\text{TSDP}}$
- Network feasible sets: $\mathcal{N}_t$ ➡ $\mathcal{N}_t^{\text{SDP}}$ $\mathcal{N}_t^{\text{TSDP}}$ $\mathcal{N}_t^{\text{SOCP}}$

⬇ ($\mathcal{N}_t^{\text{TSDP}}$)

The proposed relaxation

Relaxation of Unit Constraints

- Non-convex constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (x_{g,t} - x_{g,t-1} x_{g,t}) + \gamma_g^{\downarrow} \, (x_{g,t-1} - x_{g,t-1} x_{g,t})$$

Relaxation of Unit Constraints

- Non-convex constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (x_{g,t} - x_{g,t-1} x_{g,t}) + \gamma_g^{\downarrow} \, (x_{g,t-1} - x_{g,t-1} x_{g,t})$$

Relaxation of Unit Constraints

- Non-convex constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (x_{g,t} - x_{g,t-1} x_{g,t}) + \gamma_g^{\downarrow} \, (x_{g,t-1} - x_{g,t-1} x_{g,t})$$

- **Lifting:** Define a number of auxiliary variables:

$$o_{g,t} \triangleq p_{g,t}^2,$$

$$u_{g,t} \triangleq x_{g,t-1} x_{g,t}$$

$$y_{g,t} \triangleq p_{g,t-1} x_{g,t};$$

$$z_{g,t} \triangleq x_{g,t-1} p_{g,t};$$

Relaxation of Unit Constraints

- Non-convex constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (x_{g,t} - x_{g,t-1} x_{g,t}) + \gamma_g^{\downarrow} \, (x_{g,t-1} - x_{g,t-1} x_{g,t})$$

- **Lifting:** Define a number of auxiliary variables:

$$o_{g,t} \triangleq p_{g,t}^2, \qquad y_{g,t} \triangleq p_{g,t-1} x_{g,t},$$

$$u_{g,t} \triangleq x_{g,t-1} x_{g,t} \qquad z_{g,t} \triangleq x_{g,t-1} p_{g,t},$$

- The new variables $o_{g,t}$ and $u_{g,t}$ are defined to linearize the non-convex constraint.

Relaxation of Unit Constraints

- Non-convex constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g\, p_{g,t}^2 + \beta_g\, p_{g,t} + \gamma_g\, x_{g,t} + \gamma_g^{\uparrow}\,(x_{g,t} - x_{g,t-1}x_{g,t}) + \gamma_g^{\downarrow}\,(x_{g,t-1} - x_{g,t-1}x_{g,t})$$

$$c_{g,t} = \alpha_g\, o_{g,t} + \beta_g\, p_{g,t} + \gamma_g\, x_{g,t} + \gamma_g^{\uparrow}\,(x_{g,t} - u_{g,t}) + \gamma_g^{\downarrow}\,(x_{g,t-1} - u_{g,t})$$

- **Lifting:** Define a number of auxiliary variables:

$$o_{g,t} \triangleq p_{g,t}^2, \qquad y_{g,t} \triangleq p_{g,t-1}x_{g,t};$$

$$u_{g,t} \triangleq x_{g,t-1}x_{g,t} \qquad z_{g,t} \triangleq x_{g,t-1}p_{g,t};$$

- The new variables $o_{g,t}$ and $u_{g,t}$ are defined to linearize the non-convex constraint.

Relaxation of Unit Constraints

- Non-convex constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g\, p_{g,t}^2 + \beta_g\, p_{g,t} + \gamma_g\, x_{g,t} + \gamma_g^{\uparrow}\,(x_{g,t} - x_{g,t-1}x_{g,t}) + \gamma_g^{\downarrow}\,(x_{g,t-1} - x_{g,t-1}x_{g,t})$$

$$\Downarrow$$

$$c_{g,t} = \alpha_g\, o_{g,t} + \beta_g\, p_{g,t} + \gamma_g\, x_{g,t} + \gamma_g^{\uparrow}\,(x_{g,t} - u_{g,t}) + \gamma_g^{\downarrow}\,(x_{g,t-1} - u_{g,t})$$

- **Lifting:** Define a number of auxiliary variables:

$$o_{g,t} \triangleq p_{g,t}^2, \qquad y_{g,t} \triangleq p_{g,t-1}x_{g,t},$$

$$u_{g,t} \triangleq x_{g,t-1}x_{g,t} \qquad z_{g,t} \triangleq x_{g,t-1}p_{g,t},$$

- The new variables $o_{g,t}$ and $u_{g,t}$ are defined to linearize the non-convex constraint.
- Whereas $y_{g,t}$ and $z_{g,t}$ are used to create a stronger relaxation.

Relaxation of Unit Constraints

- Non-convex constraints:

$$x_{g,t} \in \{0,1\}$$

$$c_{g,t} = \alpha_g \, p_{g,t}^2 + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (x_{g,t} - x_{g,t-1}x_{g,t}) + \gamma_g^{\downarrow} \, (x_{g,t-1} - x_{g,t-1}x_{g,t})$$

$$c_{g,t} = \alpha_g \, o_{g,t} + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (x_{g,t} - u_{g,t}) + \gamma_g^{\downarrow} \, (x_{g,t-1} - u_{g,t})$$

- **Lifting:** Define a number of auxiliary variables:

$$o_{g,t} \triangleq p_{g,t}^2, \qquad y_{g,t} \triangleq p_{g,t-1}x_{g,t},$$

$$u_{g,t} \triangleq x_{g,t-1}x_{g,t} \qquad z_{g,t} \triangleq x_{g,t-1}p_{g,t},$$

- The new variables $o_{g,t}$ and $u_{g,t}$ are defined to linearize the non-convex constraint.
- Whereas $y_{g,t}$ and $z_{g,t}$ are used to create a stronger relaxation.
- The goal is to design a set of linear and conic inequalities that partially describe the convex hull of all feasible variables:

$$x_{g,t-2},\ x_{g,t-1},\ x_{g,t}, \quad u_{g,t-1},\ u_{g,t}, \quad p_{g,t-1},\ p_{g,t}, \quad o_{g,t-1},\ o_{g,t}, \quad y_{g,t}, \quad z_{g,t-1},\ z_{g,t}$$

Relaxation of Unit Constraints

- Linearized version of the cost equations:

$$c_{g,t} = \alpha_g \, o_{g,t} + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (x_{g,t} - u_{g,t}) + \gamma_g^{\downarrow} \, (x_{g,t-1} - u_{g,t})$$

Relaxation of Unit Constraints

- Linearized version of the cost equations:

$$c_{g,t} = \alpha_g \; o_{g,t} + \beta_g \; p_{g,t} + \gamma_g \; x_{g,t} + \gamma_g^{\uparrow} \; (x_{g,t} - u_{g,t}) + \gamma_g^{\downarrow} \; (x_{g,t-1} - u_{g,t})$$

- Binary requirement can be captured through the “McCormick constraints”:

$$\max\{0, x_{g,t-1} + x_{g,t} - 1\} \leq u_{g,t} \leq \min\{x_{g,t-1}, x_{g,t}\}$$

Relaxation of Unit Constraints

- Linearized version of the cost equations:

$$c_{g,t} = \alpha_g \, o_{g,t} + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (x_{g,t} - u_{g,t}) + \gamma_g^{\downarrow} \, (x_{g,t-1} - u_{g,t})$$

- Binary requirement can be captured through the “McCormick constraints”:

$$\max\{0, x_{g,t-1} + x_{g,t} - 1\} \leq u_{g,t} \leq \min\{x_{g,t-1}, x_{g,t}\}$$

- 2 third-order Semidefinite (TSDP) inequalities:

$$\begin{bmatrix} x_{g,t} & u_{g,t} & y_{g,t} \\ u_{g,t} & x_{g,t-1} & p_{g,t-1} \\ y_{g,t} & p_{g,t-1} & o_{g,t-1} \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} x_{g,t-1} & u_{g,t} & z_{g,t} \\ u_{g,t} & x_{g,t} & p_{g,t} \\ z_{g,t} & p_{g,t} & o_{g,t} \end{bmatrix} \succeq 0, \qquad \forall (g,t) \in \mathcal{G} \times \mathcal{T}$$

Relaxation of Unit Constraints

- Linearized version of the cost equations:

$$c_{g,t} = \alpha_g\, o_{g,t} + \beta_g\, p_{g,t} + \gamma_g\, x_{g,t} + \gamma_g^{\uparrow}\,(x_{g,t} - u_{g,t}) + \gamma_g^{\downarrow}\,(x_{g,t-1} - u_{g,t})$$

- Binary requirement can be captured through the “McCormick constraints”:

$$\max\{0, x_{g,t-1} + x_{g,t} - 1\} \leq u_{g,t} \leq \min\{x_{g,t-1}, x_{g,t}\}$$

- 2 third-order Semidefinite (TSDP) inequalities:

$$\begin{bmatrix} x_{g,t} & u_{g,t} & y_{g,t} \\ u_{g,t} & x_{g,t-1} & p_{g,t-1} \\ y_{g,t} & p_{g,t-1} & o_{g,t-1} \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} x_{g,t-1} & u_{g,t} & z_{g,t} \\ u_{g,t} & x_{g,t} & p_{g,t} \\ z_{g,t} & p_{g,t} & o_{g,t} \end{bmatrix} \succeq 0, \qquad \forall (g,t) \in \mathcal{G} \times \mathcal{T}$$

- 24 linear inequalities:

$$\left(\begin{bmatrix} & +1 & \\ 1 & -1 & \\ & & +1 \\ 1 & & -1 \end{bmatrix} \otimes \begin{bmatrix} & -\underline{p}_g & & +1 & \\ & +\bar{p}_g & & -1 & \\ & & -\underline{p}_g & & +1 \\ & & +\bar{p}_g & & -1 \\ s_g & r_g - s_g & & +1 & -1 \\ s_g & & r_g - s_g & -1 & +1 \end{bmatrix} \right) \times \begin{bmatrix} \mathbf{e}_1^\top \\ \mathbf{e}_2^\top + \mathbf{e}_6^\top + \mathbf{e}_7^\top \\ \mathbf{e}_3^\top + \mathbf{e}_{11}^\top + \mathbf{e}_{13}^\top \\ \mathbf{e}_4^\top + \mathbf{e}_9^\top \\ \mathbf{e}_5^\top + \mathbf{e}_{15}^\top \\ \mathbf{e}_8^\top + \mathbf{e}_{12}^\top \\ \mathbf{e}_{14}^\top \\ \mathbf{e}_{10}^\top \end{bmatrix}^\top \times \begin{bmatrix} 1 \\ x_{g,t-1} \\ x_{g,t} \\ p_{g,t-1} \\ p_{g,t} \\ u_{g,t} \\ y_{g,t} \\ z_{g,t} \end{bmatrix} \geq 0, \qquad \forall (g,t) \in \mathcal{G} \times \mathcal{T}$$

Relaxation of Unit Constraints

- Linearized version of the cost equations:

$$c_{g,t} = \alpha_g\, o_{g,t} + \beta_g\, p_{g,t} + \gamma_g\, x_{g,t} + \gamma_g^{\uparrow}\,(x_{g,t} - u_{g,t}) + \gamma_g^{\downarrow}\,(x_{g,t-1} - u_{g,t})$$

- Binary requirement can be captured through the "McCormick constraints":

$$\max\{0, x_{g,t-1} + x_{g,t} - 1\} \leq u_{g,t} \leq \min\{x_{g,t-1}, x_{g,t}\}$$

- 2 third-order Semidefinite (TSDP) inequalities:

$$\begin{bmatrix} x_{g,t} & u_{g,t} & y_{g,t} \\ u_{g,t} & x_{g,t-1} & p_{g,t-1} \\ y_{g,t} & p_{g,t-1} & o_{g,t-1} \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} x_{g,t-1} & u_{g,t} & z_{g,t} \\ u_{g,t} & x_{g,t} & p_{g,t} \\ z_{g,t} & p_{g,t} & o_{g,t} \end{bmatrix} \succeq 0, \qquad \forall (g,t) \in \mathcal{G} \times \mathcal{T}$$

- 24 linear inequalities:

$$\left(\begin{bmatrix} & +1 & \\ 1 & -1 & \\ & & +1 \\ 1 & & -1 \end{bmatrix} \otimes \begin{bmatrix} & -\underline{p}_g & & +1 & \\ & +\bar{p}_g & & -1 & \\ & & -\underline{p}_g & & +1 \\ & & +\bar{p}_g & & -1 \\ s_g & r_g - s_g & & +1 & -1 \\ s_g & & r_g - s_g & -1 & +1 \end{bmatrix} \right) \times \begin{bmatrix} \mathbf{e}_1^\top \\ \mathbf{e}_2^\top + \mathbf{e}_6^\top + \mathbf{e}_7^\top \\ \mathbf{e}_3^\top + \mathbf{e}_{11}^\top + \mathbf{e}_{13}^\top \\ \mathbf{e}_4^\top + \mathbf{e}_9^\top \\ \mathbf{e}_5^\top + \mathbf{e}_{15}^\top \\ \mathbf{e}_8^\top + \mathbf{e}_{12}^\top \\ \mathbf{e}_{14}^\top \\ \mathbf{e}_{10}^\top \end{bmatrix}^\top \times \begin{bmatrix} 1 \\ x_{g,t-1} \\ x_{g,t} \\ p_{g,t-1} \\ p_{g,t} \\ u_{g,t} \\ y_{g,t} \\ z_{g,t} \end{bmatrix} \geq 0, \qquad \forall (g,t) \in \mathcal{G} \times \mathcal{T}$$

Definition. *For each $g \in \mathcal{G}$, define $\mathcal{U}_g^{\mathrm{TSDP}} \subset \mathbb{R}^{T \times 4}$ to be the set of all quadruplets $(\mathbf{x}_{g,:}, \mathbf{p}_{g,:}, \mathbf{q}_{g,:}, \mathbf{c}_{g,:})$, for which there exists $(\mathbf{u}_{g,:}, \mathbf{y}_{g,:}, \mathbf{z}_{g,:}, \mathbf{o}_{g,:}) \in \mathbb{R}^{T \times 4}$, such that for every $t \in \mathcal{T}$, the above constraints hold true.*

Relaxation of Unit Constraints

- Linearized version of the cost equations:

$$c_{g,t} = \alpha_g \, o_{g,t} + \beta_g \, p_{g,t} + \gamma_g \, x_{g,t} + \gamma_g^{\uparrow} \, (x_{g,t} - u_{g,t}) + \gamma_g^{\downarrow} \, (x_{g,t-1} - u_{g,t})$$

- Binary requirement can be captured through the "McCormick constraints":

$$\max\{0, x_{g,t-1} + x_{g,t} - 1\} \leq u_{g,t} \leq \min\{x_{g,t-1}, x_{g,t}\}$$

- 2 third-order Semidefinite (TSDP) inequalities:

$$\begin{bmatrix} x_{g,t} & u_{g,t} & y_{g,t} \\ u_{g,t} & x_{g,t-1} & p_{g,t-1} \\ y_{g,t} & p_{g,t-1} & o_{g,t-1} \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} x_{g,t-1} & u_{g,t} & z_{g,t} \\ u_{g,t} & x_{g,t} & p_{g,t} \\ z_{g,t} & p_{g,t} & o_{g,t} \end{bmatrix} \succeq 0, \qquad \forall (g,t) \in \mathcal{G} \times \mathcal{T}$$

- 24 linear inequalities:

$$\left(\begin{bmatrix} & +1 & \\ 1 & -1 & \\ & & +1 \\ 1 & & -1 \end{bmatrix} \otimes \begin{bmatrix} & -\underline{p}_g & & +1 & \\ & +\bar{p}_g & & -1 & \\ & & -\underline{p}_g & & +1 \\ & & +\bar{p}_g & & -1 \\ s_g & r_g - s_g & & +1 & -1 \\ s_g & & r_g - s_g & -1 & +1 \end{bmatrix} \right) \times \begin{bmatrix} \mathbf{e}_1^\top \\ \mathbf{e}_2^\top + \mathbf{e}_6^\top + \mathbf{e}_7^\top \\ \mathbf{e}_3^\top + \mathbf{e}_{11}^\top + \mathbf{e}_{13}^\top \\ \mathbf{e}_4^\top + \mathbf{e}_9^\top \\ \mathbf{e}_5^\top + \mathbf{e}_{15}^\top \\ \mathbf{e}_8^\top + \mathbf{e}_{12}^\top \\ \mathbf{e}_{14}^\top \\ \mathbf{e}_{10}^\top \end{bmatrix}^\top \times \begin{bmatrix} 1 \\ x_{g,t-1} \\ x_{g,t} \\ p_{g,t-1} \\ p_{g,t} \\ u_{g,t} \\ y_{g,t} \\ z_{g,t} \end{bmatrix} \geq 0, \qquad \forall (g,t) \in \mathcal{G} \times \mathcal{T}$$

Definition. *For each* $g \in \mathcal{G}$*, define* $\mathcal{U}_g^{\text{TSDP}} \subset \mathbb{R}^{T \times 4}$ *to be the set of all quadruplets* $(\mathbf{x}_{g,:}, \mathbf{p}_{g,:}, \mathbf{q}_{g,:}, \mathbf{c}_{g,:})$*, for which there exists* $(\mathbf{u}_{g,:}, \mathbf{y}_{g,:}, \mathbf{z}_{g,:}, \mathbf{o}_{g,:}) \in \mathbb{R}^{T \times 4}$*, such that for every* $t \in \mathcal{T}$*, the above constraints hold true.*

Relaxation of Unit Constraints

- 4 additional inequalities can be designed in terms of the minimum down time constraints:

$$\text{If } m_g^{\downarrow} \geq 2 \quad \Rightarrow \quad \left(\begin{bmatrix} & +1 & \\ 1 & -1 & \\ & +\bar{p}_g & -1 \\ & -\underline{p}_g & +1 \end{bmatrix} \otimes \begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \end{bmatrix}^{\top} \right) \times \begin{bmatrix} \dot{\mathbf{e}}_1^{\top} \\ \dot{\mathbf{e}}_2^{\top} \\ \dot{\mathbf{e}}_3^{\top} + \dot{\mathbf{e}}_5^{\top} + \dot{\mathbf{e}}_7^{\top} \\ \dot{\mathbf{e}}_4^{\top} \\ \dot{\mathbf{e}}_6^{\top} \\ \dot{\mathbf{e}}_8^{\top} \\ \dot{\mathbf{e}}_9^{\top} + \dot{\mathbf{e}}_{11}^{\top} \\ \dot{\mathbf{e}}_{10}^{\top} \\ \dot{\mathbf{e}}_{12}^{\top} \end{bmatrix}^{\top} \times \begin{bmatrix} 1 \\ x_{g,t-2} \\ x_{g,t-1} \\ x_{g,t} \\ u_{g,t-1} \\ u_{g,t} \\ p_{g,t-1} \\ z_{g,t-1} \\ y_{g,t} \end{bmatrix} \geq 0$$

Definition. *For each $g \in \mathcal{G}$, define $\mathcal{U}_g^{\mathrm{TSDP}} \subset \mathbb{R}^{T\times 4}$ to be the set of all quadruplets $(\mathbf{x}_{g,:}, \mathbf{p}_{g,:}, \mathbf{q}_{g,:}, \mathbf{c}_{g,:})$, for which there exists $(\mathbf{u}_{g,:}, \mathbf{y}_{g,:}, \mathbf{z}_{g,:}, \mathbf{o}_{g,:}) \in \mathbb{R}^{T\times 4}$, such that for every $t \in \mathcal{T}$, the above constraints hold true.*

Relaxation of Unit Constraints

- 4 additional inequalities can be designed in terms of the minimum down time constraints:

$$\text{If } m_g^{\downarrow} \geq 2 \quad \Rightarrow \quad \left(\begin{bmatrix} & +1 & \\ 1 & -1 & \\ & +\bar{p}_g & -1 \\ & -\underline{p}_g & +1 \end{bmatrix} \otimes \begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \end{bmatrix}^{\top} \right) \times \begin{bmatrix} \dot{\mathbf{e}}_1^{\top} \\ \dot{\mathbf{e}}_2^{\top} \\ \dot{\mathbf{e}}_3^{\top} + \dot{\mathbf{e}}_5^{\top} + \dot{\mathbf{e}}_7^{\top} \\ \dot{\mathbf{e}}_4^{\top} \\ \dot{\mathbf{e}}_6^{\top} \\ \dot{\mathbf{e}}_8^{\top} \\ \dot{\mathbf{e}}_9^{\top} + \dot{\mathbf{e}}_{11}^{\top} \\ \dot{\mathbf{e}}_{10}^{\top} \\ \dot{\mathbf{e}}_{12}^{\top} \end{bmatrix}^{\top} \times \begin{bmatrix} 1 \\ x_{g,t-2} \\ x_{g,t-1} \\ x_{g,t} \\ u_{g,t-1} \\ u_{g,t} \\ p_{g,t-1} \\ z_{g,t-1} \\ y_{g,t} \end{bmatrix} \geq 0$$

- 4 additional inequalities can be designed in terms of the minimum up time constraints:

$$\text{If } m_g^{\uparrow} \geq 2 \quad \Rightarrow \quad \left(\begin{bmatrix} & +1 & \\ 1 & -1 & \\ & +\bar{p}_g & -1 \\ & -\underline{p}_g & +1 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ +1 \\ -1 \end{bmatrix}^{\top} \right) \times \begin{bmatrix} \ddot{\mathbf{e}}_1^{\top} \\ \ddot{\mathbf{e}}_2^{\top} + \ddot{\mathbf{e}}_5^{\top} \\ \ddot{\mathbf{e}}_3^{\top} \\ \ddot{\mathbf{e}}_4^{\top} \\ \ddot{\mathbf{e}}_6^{\top} \\ \ddot{\mathbf{e}}_8^{\top} \\ \ddot{\mathbf{e}}_7^{\top} \\ \ddot{\mathbf{e}}_9^{\top} \end{bmatrix}^{\top} \times \begin{bmatrix} x_{g,t-2} \\ x_{g,t-1} \\ x_{g,t} \\ u_{g,t-1} \\ u_{g,t} \\ p_{g,t-1} \\ z_{g,t-1} \\ y_{g,t} \end{bmatrix} \geq 0$$

Definition. *For each $g \in \mathcal{G}$, define $\mathcal{U}_g^{\mathrm{TSDP}} \subset \mathbb{R}^{T \times 4}$ to be the set of all quadruplets $(\mathbf{x}_{g,:}, \mathbf{p}_{g,:}, \mathbf{q}_{g,:}, \mathbf{c}_{g,:})$, for which there exists $(\mathbf{u}_{g,:}, \mathbf{y}_{g,:}, \mathbf{z}_{g,:}, \mathbf{o}_{g,:}) \in \mathbb{R}^{T \times 4}$, such that for every $t \in \mathcal{T}$, the above constraints hold true.*

Relaxation of Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \operatorname{diag}\{ \mathbf{v}_t\mathbf{v}_t^* \ \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\operatorname{diag}\{\overrightarrow{\mathbf{C}}_t \ \mathbf{v}_t\mathbf{v}_t^* \ \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\operatorname{diag}\{\overleftarrow{\mathbf{C}}_t \ \mathbf{v}_t\mathbf{v}_t^* \ \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \operatorname{diag}\{\mathbf{v}_t\mathbf{v}_t^*\} \leq \mathbf{v}_{\max}^2$$

Relaxation of Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^* \, \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^*\} \leq \mathbf{v}_{\max}^2$$

Relaxation of Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^* \, \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^*\} \leq \mathbf{v}_{\max}^2$$

$$\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^*$$

⇒

Relaxation of Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \text{diag}\{\mathbf{v}_t\mathbf{v}_t^* \, \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \text{diag}\{\mathbf{v}_t\mathbf{v}_t^*\} \leq \mathbf{v}_{\max}^2$$

$$\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^*$$

Lifted Space:

$$\mathbf{d}_t + \text{diag}\{\mathbf{W}_t \, \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t \, \mathbf{W}_t^* \, \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t \, \mathbf{W}_t^* \, \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \text{diag}\{\mathbf{W}_t\} \leq \mathbf{v}_{\max}^2$$

Relaxation of Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^* \, \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^*\} \leq \mathbf{v}_{\max}^2$$

$$\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^*$$

Lifted Space:

$$\mathbf{d}_t + \mathrm{diag}\{\mathbf{W}_t \, \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \, \mathbf{W}_t^* \, \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \, \mathbf{W}_t^* \, \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \mathrm{diag}\{\mathbf{W}_t\} \leq \mathbf{v}_{\max}^2$$

- The network constraints can be cast linearly w.r.t. $\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^*$.

Relaxation of Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \text{diag}\{\mathbf{v}_t\mathbf{v}_t^* \, \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \text{diag}\{\mathbf{v}_t\mathbf{v}_t^*\} \leq \mathbf{v}_{\max}^2$$

$$\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^* \;\Rightarrow$$

Lifted Space:

$$\mathbf{d}_t + \text{diag}\{\mathbf{W}_t \, \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t \, \mathbf{W}_t^* \, \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t \, \mathbf{W}_t^* \, \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \text{diag}\{\mathbf{W}_t\} \leq \mathbf{v}_{\max}^2$$

- The network constraints can be cast linearly w.r.t. $\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^*$.
- Semidefinite programming (SDP) relaxation: $\mathbf{W}_t \succeq 0$

Relaxation of Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \text{diag}\{\mathbf{v}_t\mathbf{v}_t^* \, \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overrightarrow{\mathbf{Y}}_t^*\}| \le \mathbf{f}_{\max;t}$$

$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t \, \mathbf{v}_t\mathbf{v}_t^* \, \overleftarrow{\mathbf{Y}}_t^*\}| \le \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \le \text{diag}\{\mathbf{v}_t\mathbf{v}_t^*\} \le \mathbf{v}_{\max}^2$$

$$\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^* \quad \Rightarrow$$

Lifted Space:

$$\mathbf{d}_t + \text{diag}\{\mathbf{W}_t \, \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t \, \mathbf{W}_t^* \, \overrightarrow{\mathbf{Y}}_t^*\}| \le \mathbf{f}_{\max;t}$$

$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t \, \mathbf{W}_t^* \, \overleftarrow{\mathbf{Y}}_t^*\}| \le \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \le \text{diag}\{\mathbf{W}_t\} \le \mathbf{v}_{\max}^2$$

- The network constraints can be cast linearly w.r.t. $\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^*$.
- Semidefinite programming (SDP) relaxation: $\mathbf{W}_t \succeq 0$
- **Decomposed SDP:** A collection of overlapping subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_A \subseteq \mathcal{V}$ are obtained from a graph-theoretic analysis of the network:

$$\mathbf{W}_t[\mathcal{A}_k, \mathcal{A}_k] \succeq 0, \qquad \forall k \in \{1, 2, \ldots, A\}$$

Relaxation of Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \text{diag}\{\mathbf{v}_t\mathbf{v}_t^* \; \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t \; \mathbf{v}_t\mathbf{v}_t^* \; \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t \; \mathbf{v}_t\mathbf{v}_t^* \; \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \text{diag}\{\mathbf{v}_t\mathbf{v}_t^*\} \leq \mathbf{v}_{\max}^2$$

$$\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^*$$

Lifted Space:

$$\mathbf{d}_t + \text{diag}\{\mathbf{W}_t \; \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\text{diag}\{\overrightarrow{\mathbf{C}}_t \; \mathbf{W}_t^* \; \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\text{diag}\{\overleftarrow{\mathbf{C}}_t \; \mathbf{W}_t^* \; \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \text{diag}\{\mathbf{W}_t\} \leq \mathbf{v}_{\max}^2$$

- The network constraints can be cast linearly w.r.t. $\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^*$.
- Semidefinite programming (SDP) relaxation: $\mathbf{W}_t \succeq 0$
- **Decomposed SDP:** A collection of overlapping subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_A \subseteq \mathcal{V}$ are obtained from a graph-theoretic analysis of the network:

$$\mathbf{W}_t[\mathcal{A}_k, \mathcal{A}_k] \succeq 0, \qquad \forall k \in \{1, 2, \ldots, A\}$$

- **Second-order cone programming (SOCP):**

$$\begin{bmatrix} W_{k_1,k_1} & W_{k_1,k_2} \\ W_{k_2,k_1} & W_{k_2,k_2} \end{bmatrix} \succeq 0, \qquad \forall (k_1, k_2) \in \mathcal{E}_t$$

Relaxation of Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^* \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \mathbf{v}_t\mathbf{v}_t^* \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \mathbf{v}_t\mathbf{v}_t^* \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^*\} \leq \mathbf{v}_{\max}^2$$

$$\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^*$$

Lifted Space:

$$\mathbf{d}_t + \mathrm{diag}\{\mathbf{W}_t \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \mathbf{W}_t^* \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \mathbf{W}_t^* \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \mathrm{diag}\{\mathbf{W}_t\} \leq \mathbf{v}_{\max}^2$$

- SDP relaxation is computationally expensive while SOCP has poor performance:

SDP Relaxation

SOCP Relaxation

Relaxation of Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^* \ \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \ \mathbf{v}_t\mathbf{v}_t^* \ \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \ \mathbf{v}_t\mathbf{v}_t^* \ \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^*\} \leq \mathbf{v}_{\max}^2$$

$$\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^* \Rightarrow$$

Lifted Space:

$$\mathbf{d}_t + \mathrm{diag}\{\mathbf{W}_t \ \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \ \mathbf{W}_t^* \ \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \ \mathbf{W}_t^* \ \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \mathrm{diag}\{\mathbf{W}_t\} \leq \mathbf{v}_{\max}^2$$

- SDP relaxation is computationally expensive while SOCP has poor performance:

SDP Relaxation | TSDP Relaxation | SOCP Relaxation

- The proposed TSDP relaxation:

$$\begin{bmatrix} W_{k_1,k_1} & W_{k_1,k_2} & W_{k_1,k_3} \\ W_{k_2,k_1} & W_{k_2,k_2} & W_{k_2,k_3} \\ W_{k_3,k_1} & W_{k_3,k_2} & W_{k_3,k_3} \end{bmatrix} \succeq 0, \qquad \forall (k_1, k_2, k_3) \in \bigcup_{k=1}^{A} \mathcal{A}_k \times \mathcal{A}_k \times \mathcal{A}_k$$

Relaxation of Network Constraints

AC Network Constraints:

$$\mathbf{d}_t + \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^* \; \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \; \mathbf{v}_t\mathbf{v}_t^* \; \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \; \mathbf{v}_t\mathbf{v}_t^* \; \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \mathrm{diag}\{\mathbf{v}_t\mathbf{v}_t^*\} \leq \mathbf{v}_{\max}^2$$

$$\mathbf{W}_t \triangleq \mathbf{v}_t\mathbf{v}_t^*$$

Lifted Space:

$$\mathbf{d}_t + \mathrm{diag}\{\mathbf{W}_t \; \mathbf{Y}_t^*\} = \mathbf{C}^\top(\mathbf{p}_t + i\mathbf{q}_t)$$

$$|\mathrm{diag}\{\overrightarrow{\mathbf{C}}_t \; \mathbf{W}_t^* \; \overrightarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$|\mathrm{diag}\{\overleftarrow{\mathbf{C}}_t \; \mathbf{W}_t^* \; \overleftarrow{\mathbf{Y}}_t^*\}| \leq \mathbf{f}_{\max;t}$$

$$\mathbf{v}_{\min}^2 \leq \mathrm{diag}\{\mathbf{W}_t\} \leq \mathbf{v}_{\max}^2$$

- SDP relaxation is computationally expensive while SOCP has poor performance:

SDP Relaxation | TSDP Relaxation | SOCP Relaxation

- The proposed TSDP relaxation:

$$\begin{bmatrix} W_{k_1,k_1} & W_{k_1,k_2} & W_{k_1,k_3} \\ W_{k_2,k_1} & W_{k_2,k_2} & W_{k_2,k_3} \\ W_{k_3,k_1} & W_{k_3,k_2} & W_{k_3,k_3} \end{bmatrix} \succeq 0, \qquad \forall (k_1,k_2,k_3) \in \bigcup_{k=1}^{A} \mathcal{A}_k \times \mathcal{A}_k \times \mathcal{A}_k$$

- More scalable than SDP and more likely to result in a feasible solution compared to SOCP.

Convex Relaxation

- Consider a problem with T time periods and G units:

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\
\text{subject to} \quad & (\mathbf{x}_{g,:}, \mathbf{p}_{g,:}, \mathbf{q}_{g,:}, \mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\
& (\mathbf{p}_{:,t}, \mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T},
\end{aligned}
$$

Convex Relaxation

- Consider a problem with T time periods and G units:

$$\begin{aligned} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ \text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g \qquad \forall g \in \mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t \qquad \forall t \in \mathcal{T}, \end{aligned}$$

- Third-order semidefinite relaxation:

$$\begin{aligned} \underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} \quad & \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ \text{subject to} \quad & (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g^{\text{TSDP}} \qquad \forall g \in \mathcal{G}, \\ & (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t^{\text{TSDP}} \qquad \forall t \in \mathcal{T}, \end{aligned}$$

Convex Relaxation

- Consider a problem with T time periods and G units:

$$\begin{aligned} &\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} && \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ &\text{subject to} && (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g && \forall g\in\mathcal{G}, \\ &&& (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t && \forall t\in\mathcal{T}, \end{aligned}$$

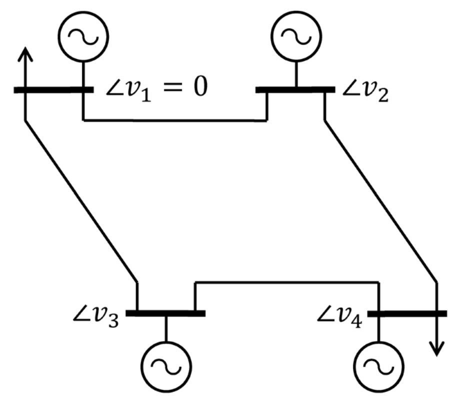
- Third-order semidefinite relaxation:

$$\begin{aligned} &\underset{\mathbf{x},\mathbf{p},\mathbf{q},\mathbf{c}\in\mathbb{R}^{G\times T}}{\text{minimize}} && \sum_{g=1}^{G}\sum_{t=1}^{T} c_{g,t} \\ &\text{subject to} && (\mathbf{x}_{g,:},\mathbf{p}_{g,:},\mathbf{q}_{g,:},\mathbf{c}_{g,:}) \in \mathcal{U}_g^{\text{TSDP}} && \forall g\in\mathcal{G}, \\ &&& (\mathbf{p}_{:,t},\mathbf{q}_{:,t}) \in \mathcal{N}_t^{\text{TSDP}} && \forall t\in\mathcal{T}, \end{aligned}$$

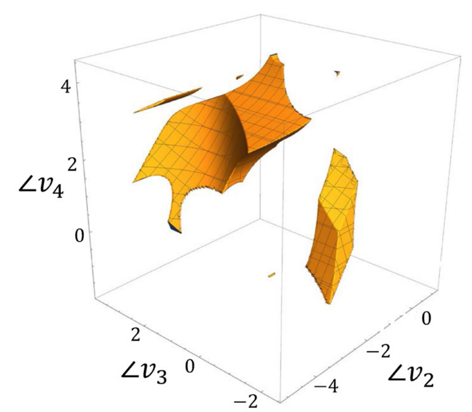
Conclusions

- Third-order semidefinite relaxation:
  - Joint UC-AC-OPF.

4-bus power network:



Feasible region of voltage angles:

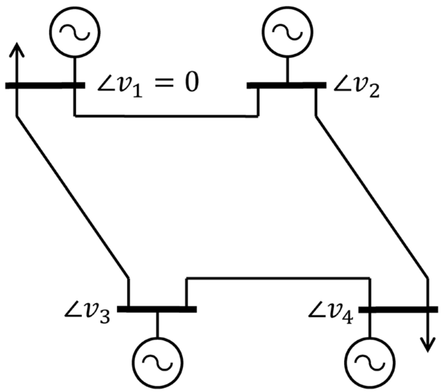


Conclusions

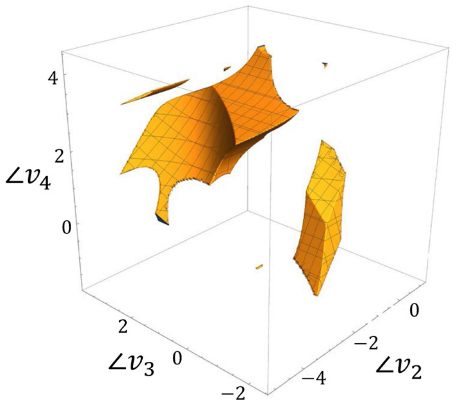
- Third-order semidefinite relaxation:
  - Joint UC-AC-OPF.

- Advantages:
  - Compatible with the AC model of networks,
  - Massively scalable,
  - Near-globally optimal feasible points.

4-bus power network:



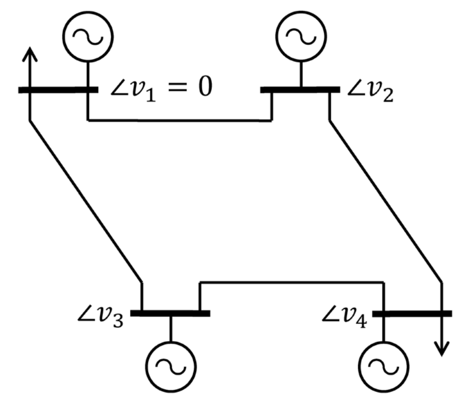
Feasible region of voltage angles:



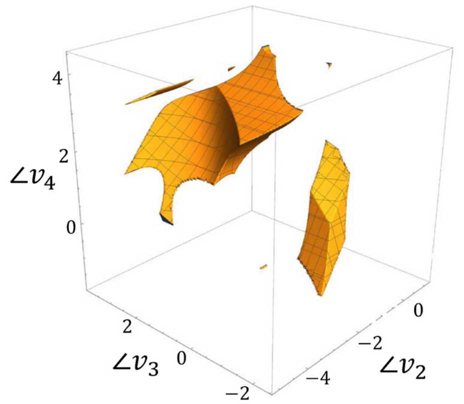
Conclusions

- Third-order semidefinite relaxation:
  - Joint UC-AC-OPF.
- Advantages:
  - Compatible with the AC model of networks,
  - Massively scalable,
  - Near-globally optimal feasible points.
- Future directions:
  - Incorporation of uncertainties and security consideration,
  - **Implementation on GPU:** Linear algebra on 3 x 3 matrices has closed-form solutions.

4-bus power network:



Feasible region of voltage angles:



Thank you