Convex Formulation of the Optimal Transmission Switching Problem

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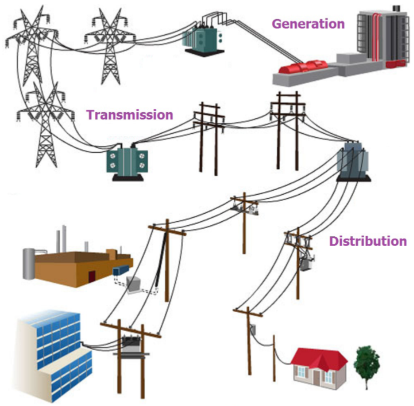
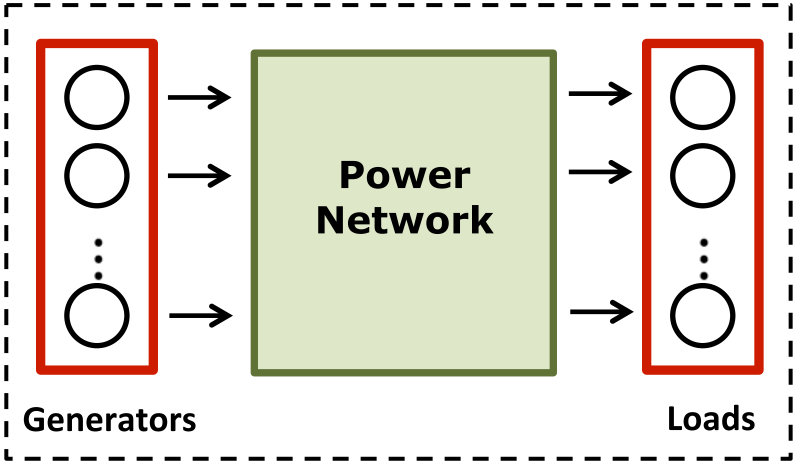
Outline

1. Introduction and problem formulation.
2. Tightening the linearized relaxation.
    - Effect on practical problems with linear objective.
3. Convex model of the problem with quadratic objective.
    - Different conic relaxations
    - Performance of the convex model in practical problems.
4. Boosting the speed using heuristic rounding schemes.

Power Systems



- **Power system:**
  - A large-scale system consisting of generators, loads, lines, etc.
  - Used for generating, transporting and distributing electricity.



Optimal Transmission Switching Problem: Find an optimal topology of the power network that minimizes the operational cost, subject to energy demand and operating constraints.

OTS Problem

- In plain language: Pick a subset of lines in the network that maximizes the performance.

- Intuition: Adding more lines increases the capacity of the network.
- Does the problem correspond to a conventional network flow problem? Min cost network flow or max flow problem.
- Answer: No!
- Key difference: We have more variables and constraints on the edges.

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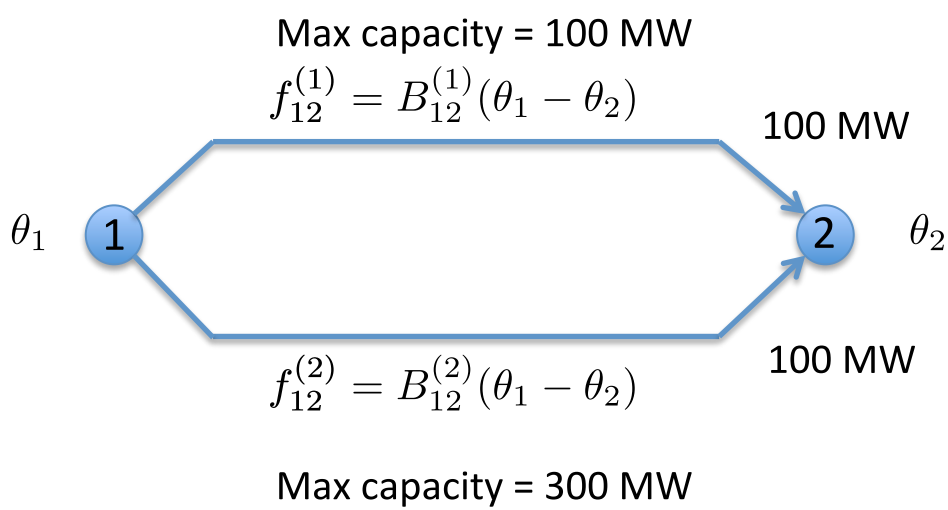

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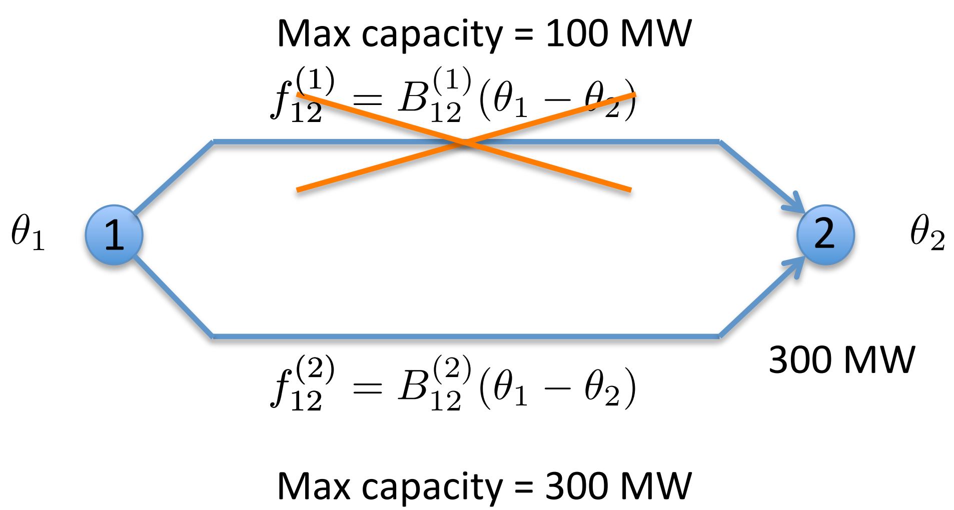

Problem Formulation

- Decision variables: $\mathbf{x} \triangleq [x_1, \ldots, x_{n_s}]^\top, \theta \triangleq [\theta_1, \ldots, \theta_{n_b}]^\top \; \mathbf{f} \triangleq [f_1, \ldots, f_{n_l}], \; \mathbf{p} \triangleq [p_1, \ldots, p_{n_g}]^\top,$

- Generation costs:
  - Power generation costs: $\sum_{i=1}^{n_g} a_i \times (p_i)^2 + b_i \times p_i \qquad \sum_{i=1}^{n_g} b_i \times p_i$

- Constraints:
  - Switch status: $x_i \in \{0, 1\}$
  - Generator limits: $p_{i;\,\min} \leq p_i \leq p_{i;\,\max}$
  - Flow limits for inflexible lines: $-f_{ij;\,\max} \leq f_{ij} \leq f_{ij;\,\max} \quad \forall (i,j) \in \mathcal{L} \backslash \mathcal{S}$
  - Flow limits for flexible lines: $-f_{ij;\,\max} \times x_{ij} \leq f_{ij} \leq f_{ij;\,\max} \times x_{ij} \quad \forall (i,j) \in \mathcal{S}$
  - Physical constraints for inflexible lines: $f_{ij} = B_{ij}(\theta_i - \theta_j) \quad (i,j) \in \mathcal{L} \backslash \mathcal{S}$
  - Physical constraints for flexible lines: $f_{ij} = B_{ij}(\theta_i - \theta_j) x_{ij} \quad (i,j) \in \mathcal{S}$
  - Conservation of flows: $p_k - d_i = \sum_{j \in \mathcal{N}_l^+(i)} f_{ij} - \sum_{j \in \mathcal{N}_l^-(i)} f_{ji}$
  - Cardinality constraint: $\sum_{(i,j) \in \mathcal{S}} x_{ij} \geq |\mathcal{L}| - r$

Linearization

- We can also add time horizon, security constraints, etc.
- OTS is an NP-hard problem.
- Nonlinear and nonconvex constraints:

$$f_{ij} = B_{ij}(\theta_i - \theta_j)x_{ij} \quad (i,j) \in \mathcal{S}$$

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Exact Linearization!

$$B_{ij}(\theta_i - \theta_j) - M_{ij}(1 - x_{ij}) \leq f_{ij} \leq B_{ij}(\theta_i - \theta_j) + M_{ij}(1 - x_{ij})$$

$$-f_{ij;max} \times x_{ij} \leq f_{ij} \leq f_{ij;max} \times x_{ij}$$

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Question: How large should M_{ij} be?

1. Small values for M_{ij} can lead to tighter relaxations and hence, fewer iterations.
2. Large values for M_{ij} can lead to numerical issues.

Definition: M_{ij} is called feasible if it results in exact linearization.

Theorem: For a flexible line (i,j):
1. There is no efficient algorithm to find the smallest feasible M_{ij}.
2. There is no efficient approximation algorithm to find the smallest feasible M_{ij}.

1. S. Fattahi, J. Lavaei, A. Atamturk, "Promises of Conic Relaxations in Optimal Transmission Switching of Power Systems", submitted for publication.

Linearization

- Small value for M_{ij} is highly desirable.
- Trivial upper bounds for feasible M_{ij}.
- Can we go from trivial to nontrivial values?
- Common practice: add upper bounds on the absolute value of angles.
- May significantly shrink the feasible region.

Observation: Only a small subset of lines are considered as flexible.

Nontrivial upper bounds can be found if there is a connected sub-network with no switches.

Simulation Results

- We consider the IEEE 118-bus system. This system has 118 nodes and 185 lines.
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  - 68 binary variables corresponding to switches.
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  - 185 continuous variables corresponding to the flow of each line.
  - 54 continuous variables corresponding to the generation of different generators.
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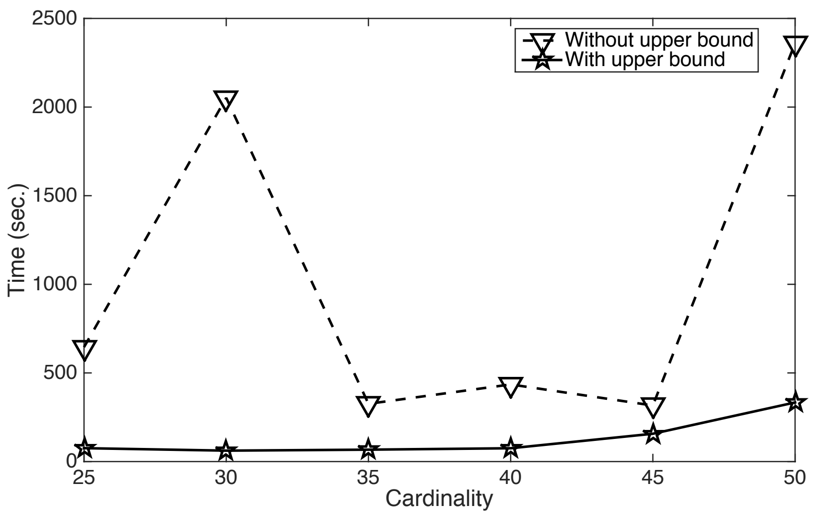
- Intel Core i7 quad-core 2.50 GHz CPU and 16GB RAM.
- Serial implementation in MATLAB using the CVX framework and Gurobi solver.
- The optimality threshold is set to 0.01.

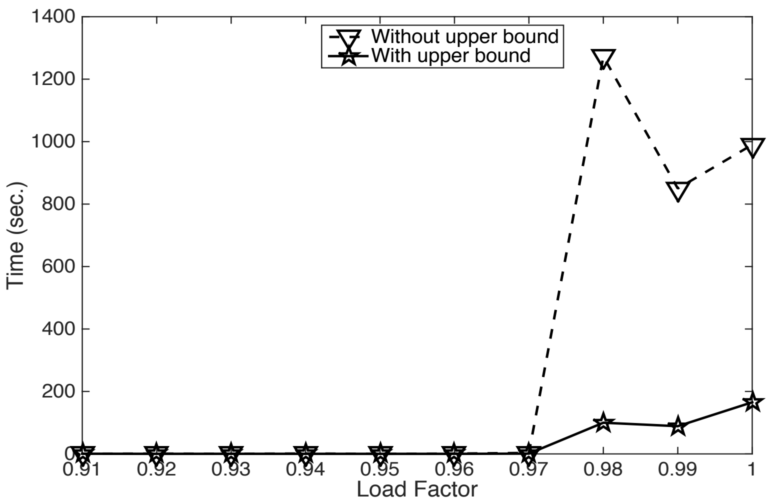
With designed upper bounds on M_{ij}	**Without designed upper bounds on** M_{ij}
2 min	46 min

Simulation Results

Performance with respect to cardinality lower bound



Performance with respect to different load factors



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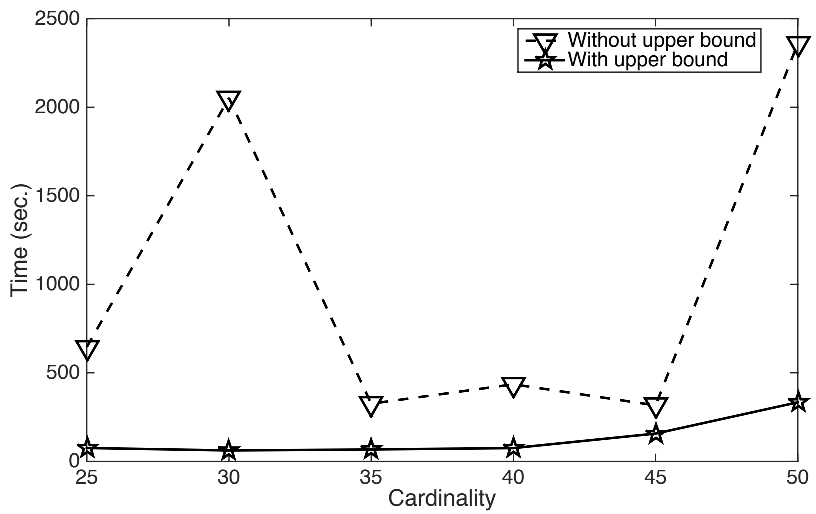

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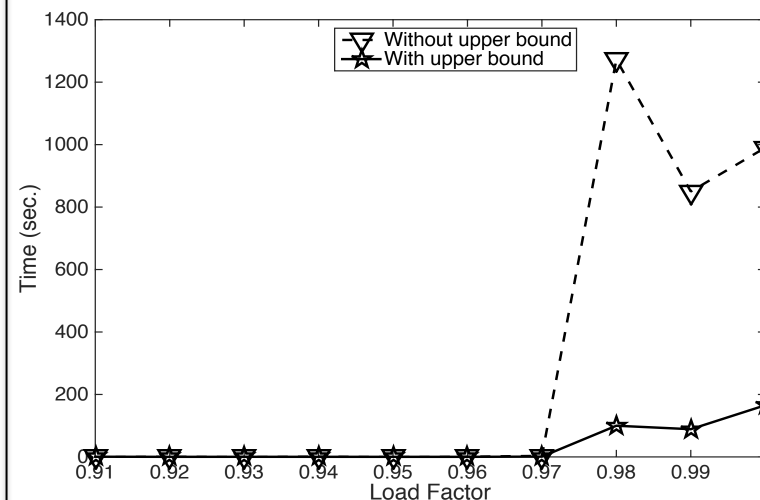

- If cardinality lower bound is zero, < 1 sec for Gurobi to solve the problem.
- Linear objective.
- If the objective function is quadratic, Gurobi finds the optimal solution after 72 min!

Convex Model

- Existing methods are based on branch and bound, cutting plane, dynamic programming, or line rankings.
- **Goal:** Find a convex model of the problem.
  - Useful for *convex hull pricing.* [Gribik 07]
  - Can be adopted to solve OTS problem for AC systems.

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- If the optimal W has rank-1, the relaxation is exact.

Bad news: SDP relaxation is almost as bad as QP.

Theorem: For generic load profiles, the SDP will work with probability 0.

- The optimality gap is 3%-80% in IEEE test cases.
- Need to strengthen the formulation by adding valid inequalities.

Valid Inequalities

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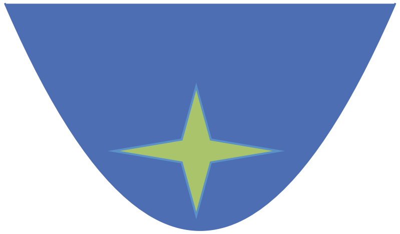

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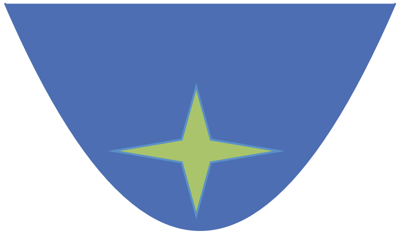
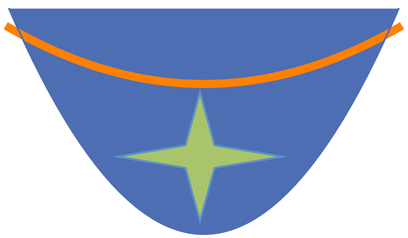


$\mathbf{u}^\top \mathbf{w} - z_1 \geq 0 \quad \mathbf{v}^\top \mathbf{w} - z_2 \geq 0$

Nonlinear constraint



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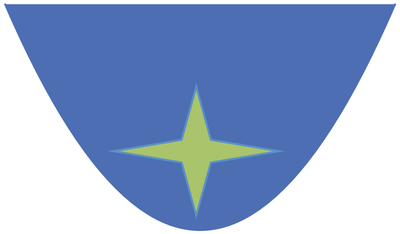

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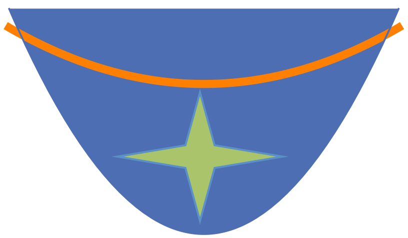
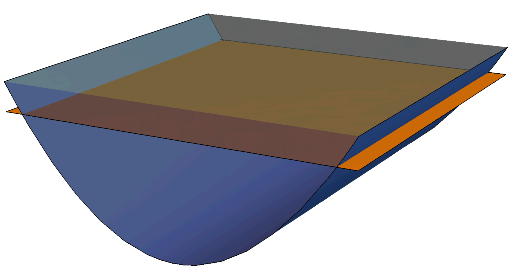


$$\mathbf{u}^\top \mathbf{w}\mathbf{w}^\top \mathbf{v} - (\mathbf{v}^\top z_1 + \mathbf{u}^\top z_2)\mathbf{w} + z_1 z_2 \geq 0$$

Linearize!



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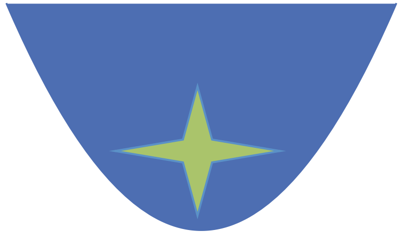

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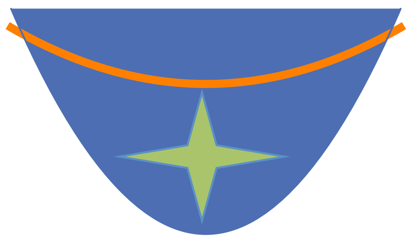

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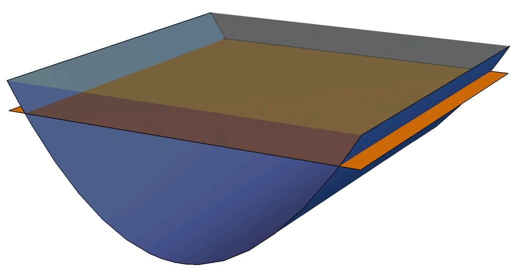

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Based on Sherali-Adams' RLT relaxation.

$$\begin{aligned}
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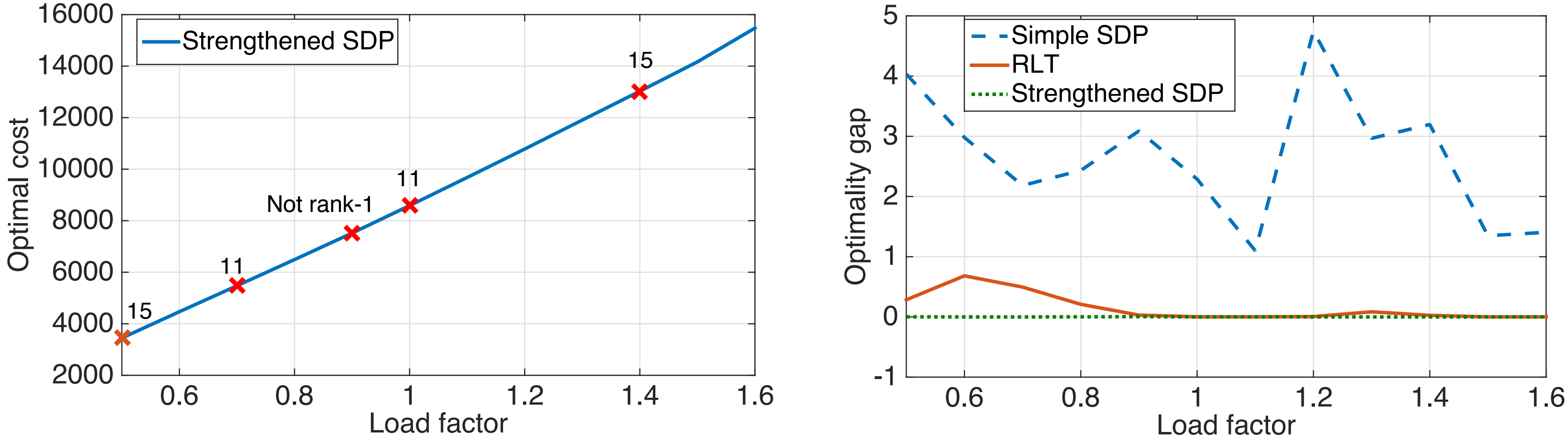
& W_{kk} - w_k = 0, \qquad k = 1, 2, \ldots, n_s, \\
& \mathbf{W} \succeq \mathbf{w}\mathbf{w}^\top.
\end{aligned}$$

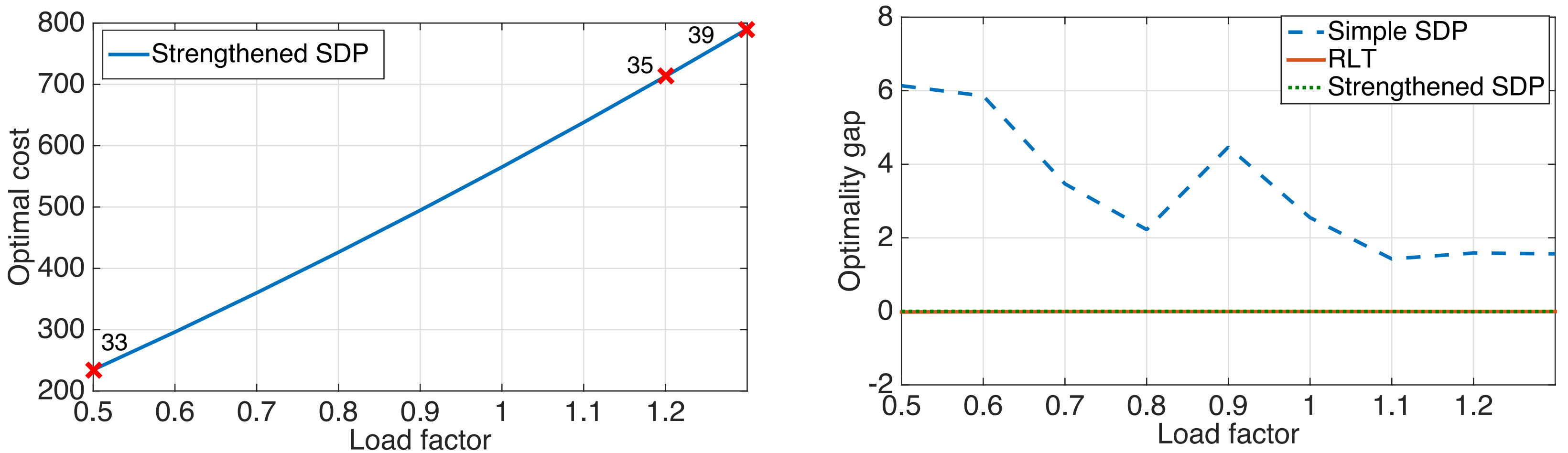
Theorem: This relaxation is exact for large loads and/or small line ratings.

Simulation Results

❑ IEEE 14-bus with 12 load scenarios and 5 switches:



❑ IEEE 30-bus with 9 load scenarios and 7 switches:



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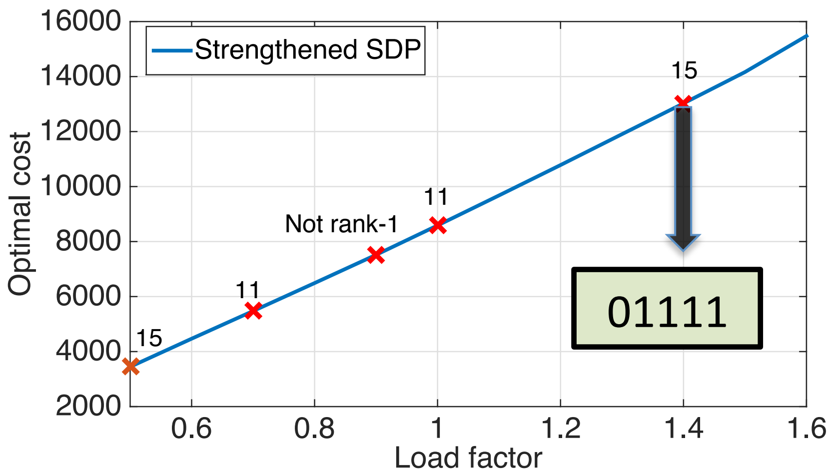

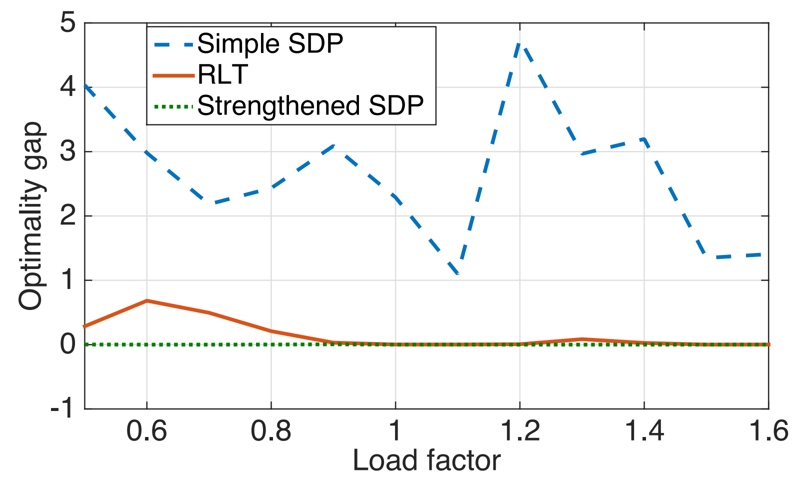

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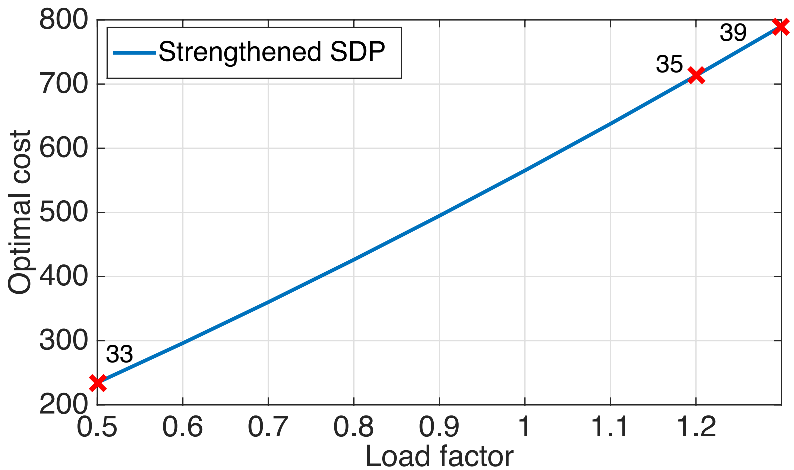
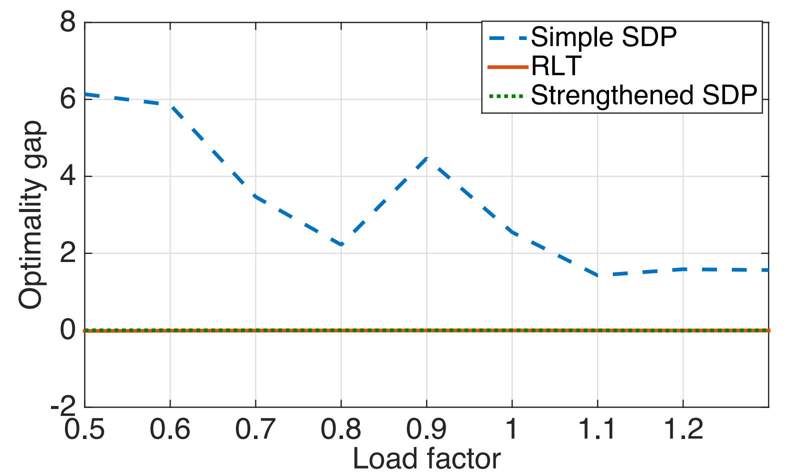
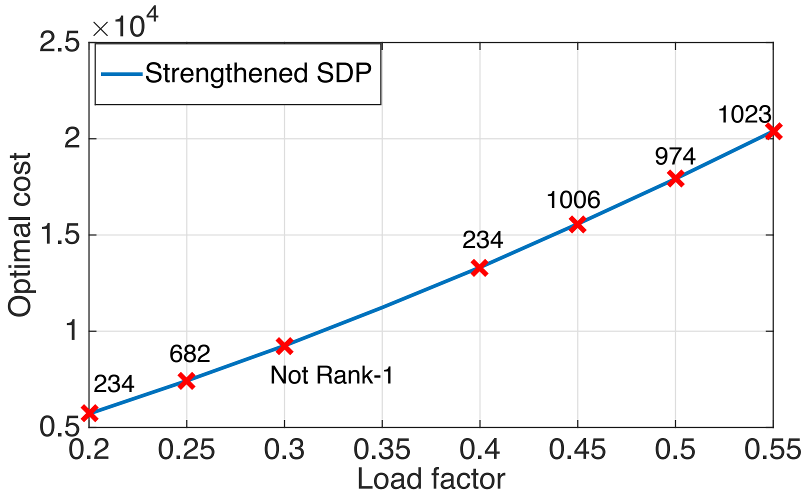
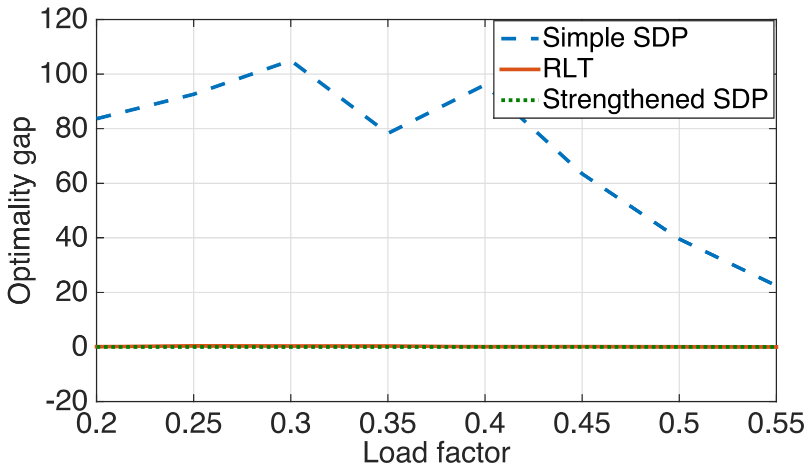




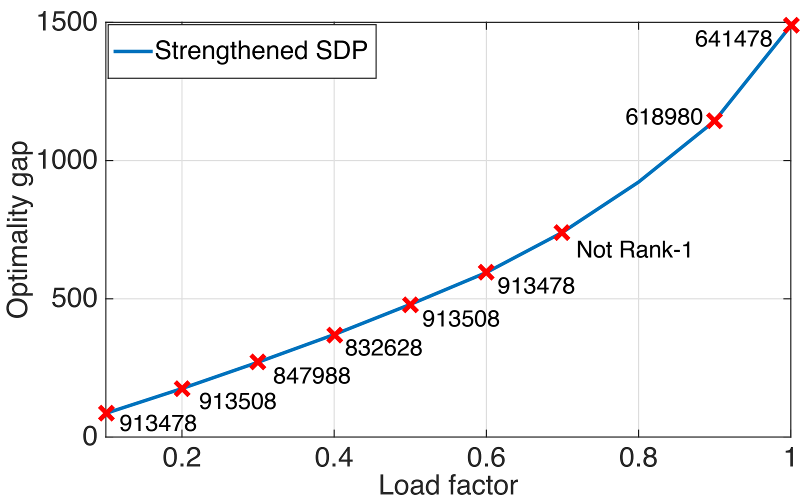
Simulation Results

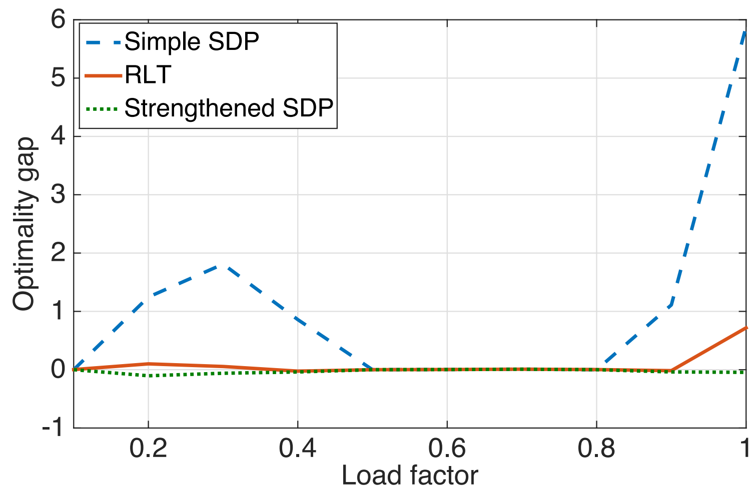
- IEEE 57-bus with 8 load scenarios and 10 switches:





- IEEE 118-bus with 10 load scenarios and 20 switches and lower bound equal to 10:





Boosting the Speed: Heuristic Rounding

- The SSDP is extremely time-consuming to solve.
- 10 min to solve IEEE 118-bus system.
- It finds the optimal objective, but not the feasible binary variables.
- We can resort to RLT with heuristic rounding.

- We consider the IEEE 118-bus system with same settings.

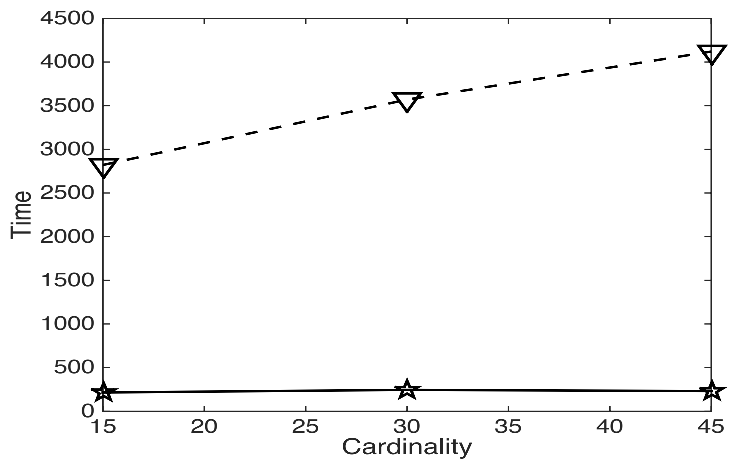
- Number of variables in RLT relaxation: 90,525
- Number of constraints in RLT relaxation: 376,333

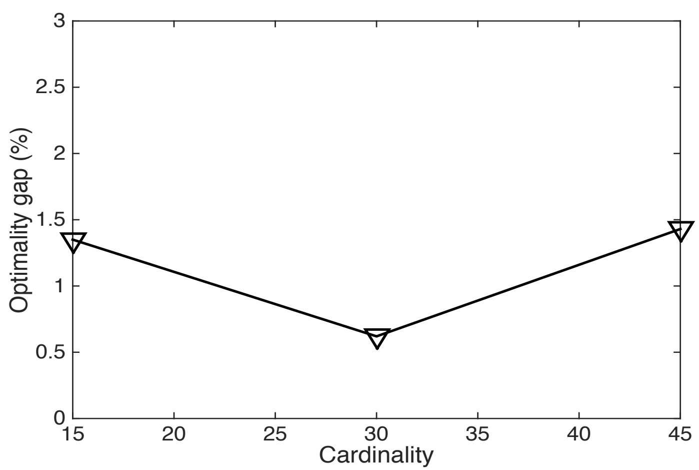
- Implementing RLT relaxation in Gurobi was extremely inefficient.
- Instead, we used MOSEK solver with CVX framework in MATLAB.

- Only 2 levels of RLT were needed in our simulations.
- Number of rounded binary variables after first round of RLT: 50/65

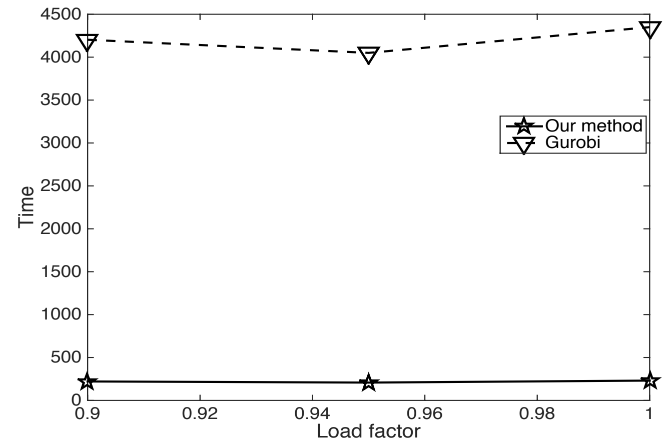
Numerical Results

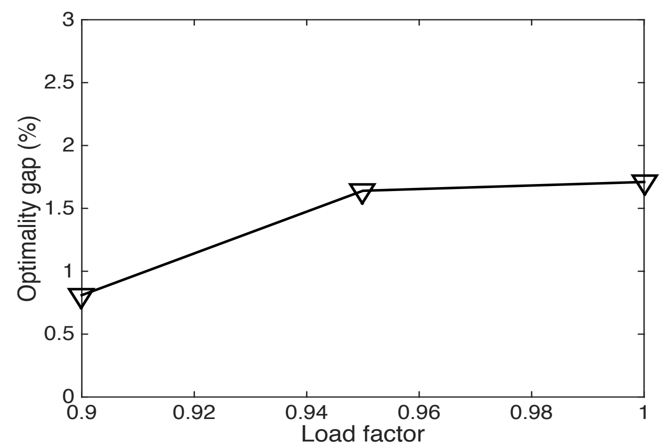
Performance with respect to cardinality lower bound





Performance with respect to different load factors





Extension to Unit Commitment

- Binary variables for generators (ON/OFF).
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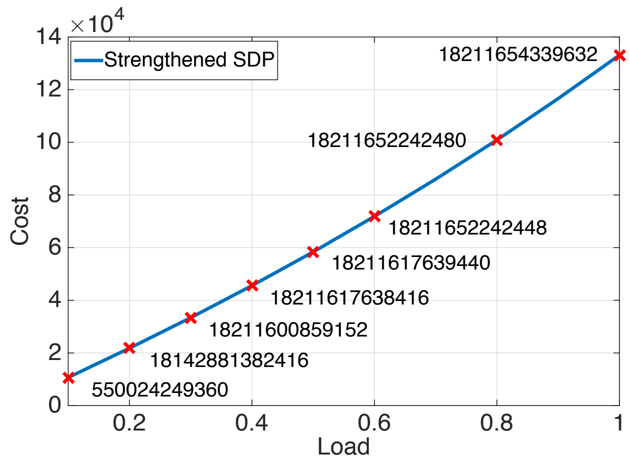
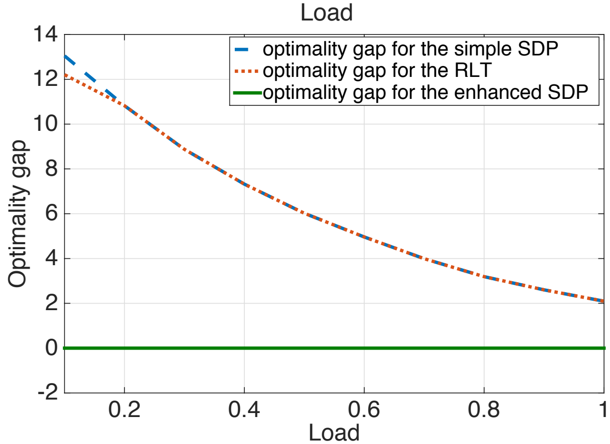
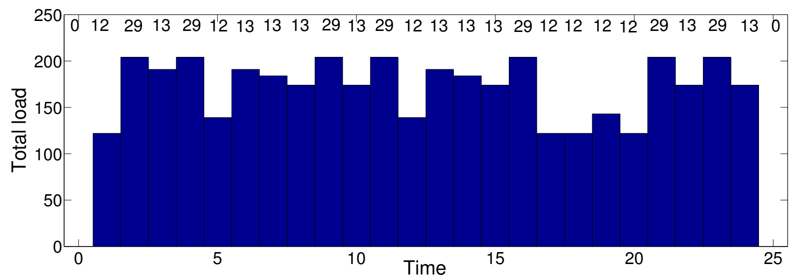
IEEE 118-bus with 54 generators:





IEEE 14-bus over 24 hours:



SDP relaxation: 162600

Reduced-strengthened SDP relaxation: 205838

Conclusions

- OTS problem with linear and quadratic objectives.
- Finding a good MILP formulation of OTS problem may be hard.
- The MILP formulation can be tightened if some part of the topology is fixed.
- Convex model for OTS problem with quadratic objective.
- Strong valid inequalities.
- Rounding heuristics in order to boost the running time.
- Extension to Unit Commitment problem.

Thank you!