Power System State Estimation with a Limited Number of Measurements

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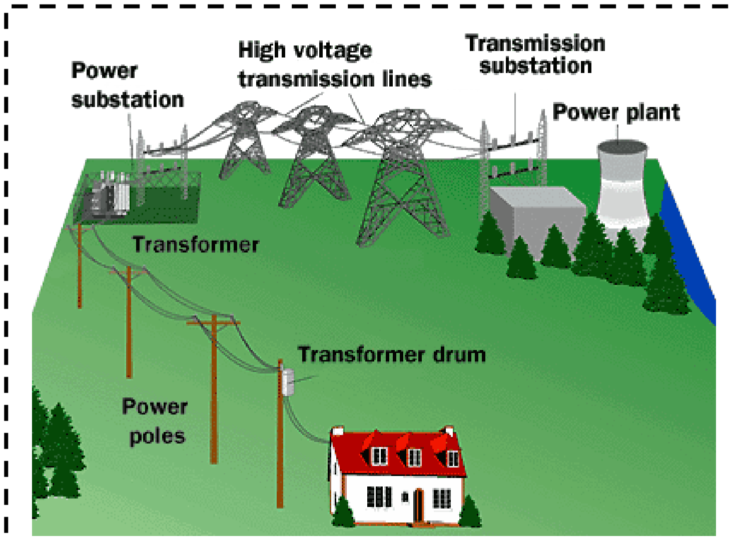
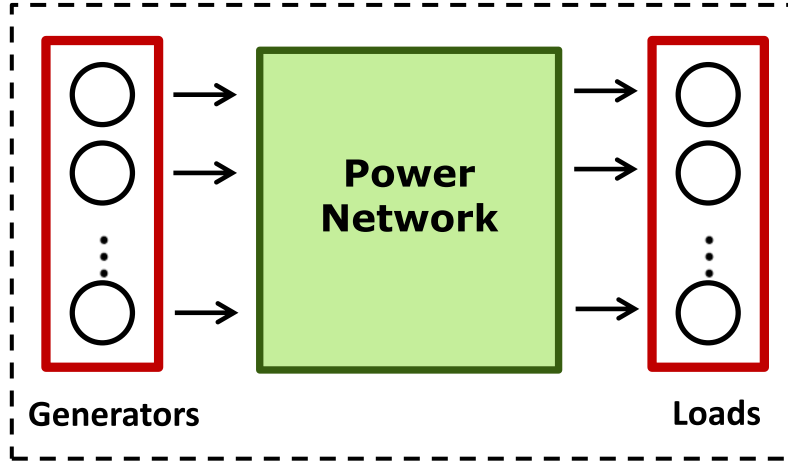
Power System State Estimation (PSSE)

Network Parameters:

1. Voltages
2. Currents
3. Phase angles
4. Power injections
5. Power flows



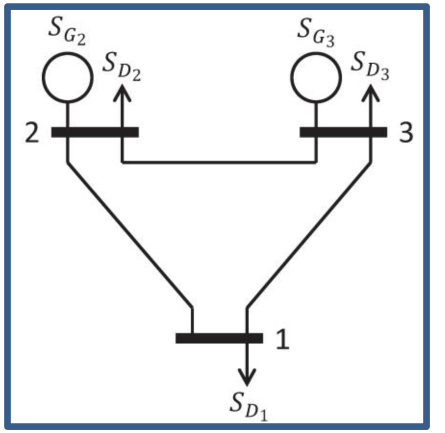


PSSE: Estimating the unknown parameters of the network, based on a limited number of measurements corrupted with noise and bad data.

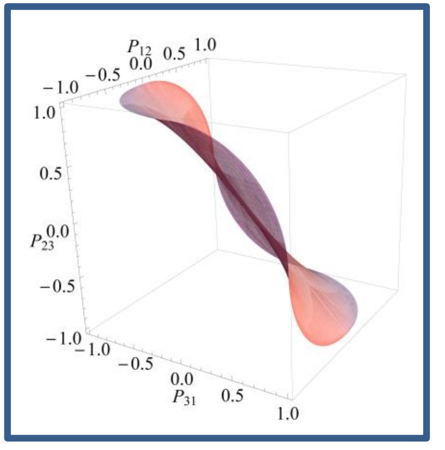
Power System State Estimation (PSSE)

Noiseless Scenario: Power flow feasibility

Simple network:



Feasible set:

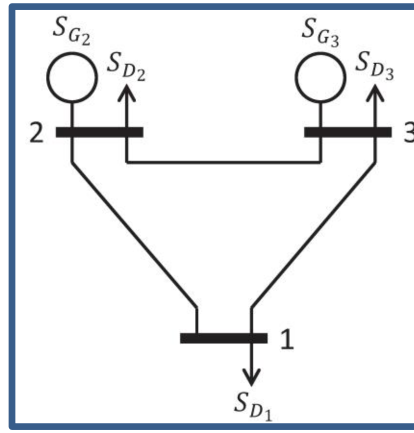


Power System State Estimation (PSSE)

Noiseless Scenario: Power flow feasibility

Nonlinear
Network Equations

Simple network:



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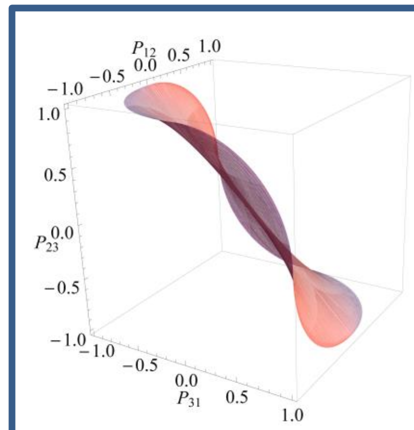

Power System State Estimation (PSSE)

Noiseless Scenario: Power flow feasibility

Nonlinear Network Equations → NP-hard

- **Complexity:** Strongly NP-hard [Bienstock et al., 09].

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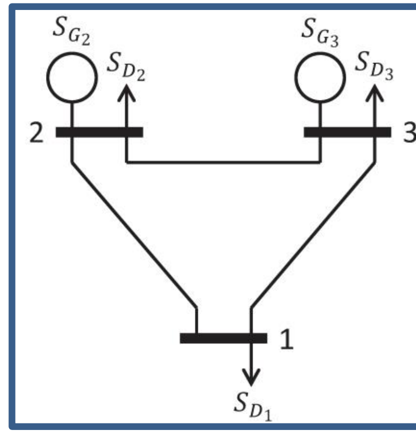

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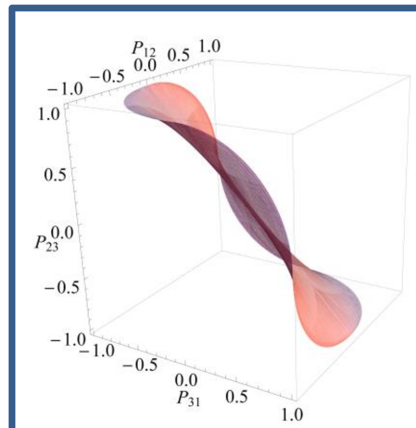

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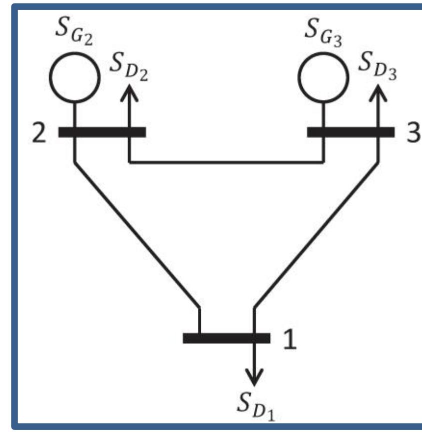

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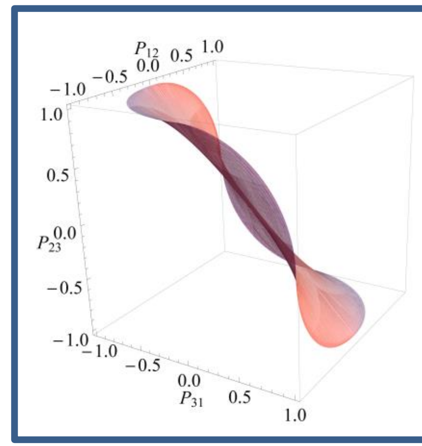

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Noiseless Scenario: Power flow feasibility

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- **Complexity:** Strongly NP-hard [Bienstock et al., 09].
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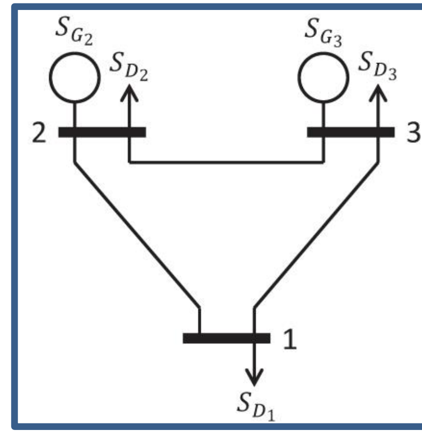

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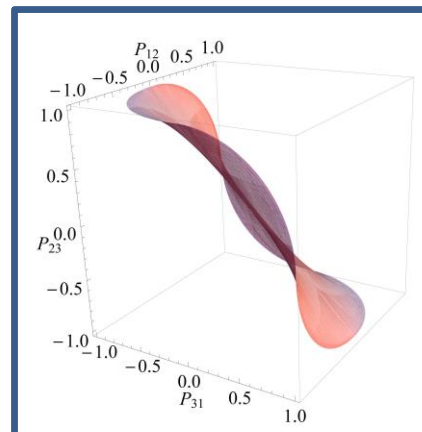

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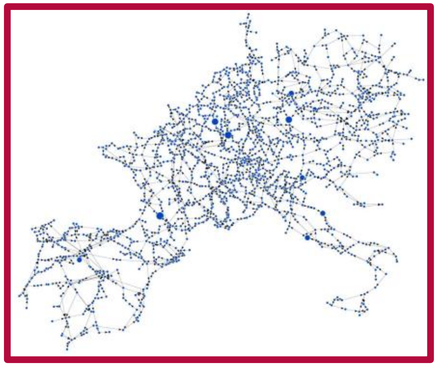

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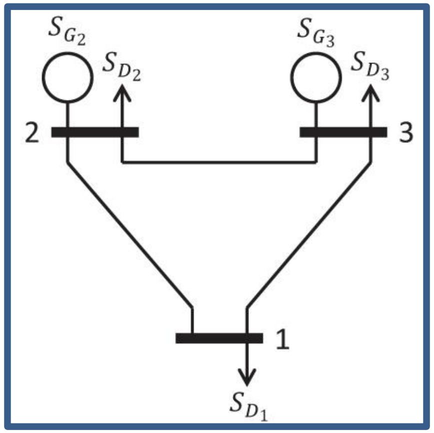

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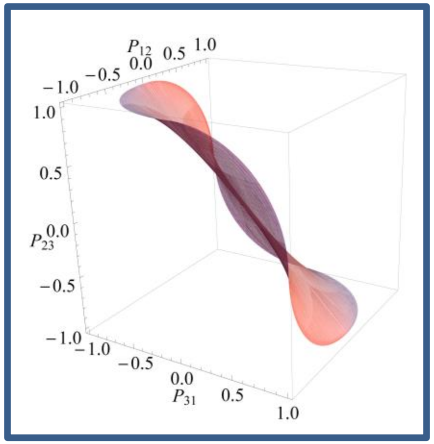

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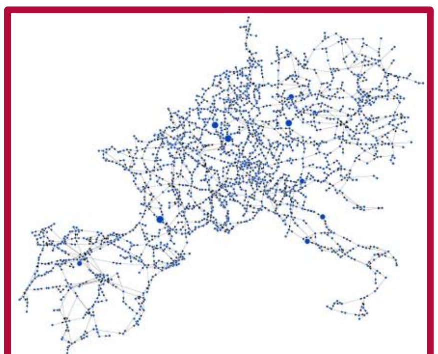
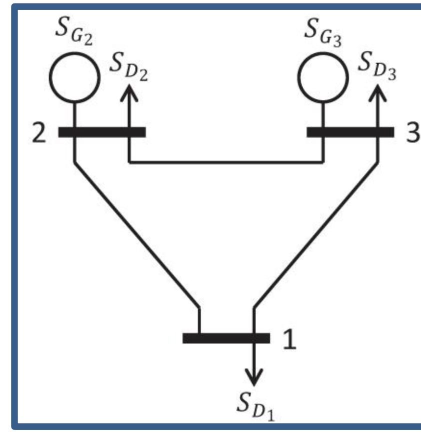


- European grid with 9241 nodes.

Simple network:



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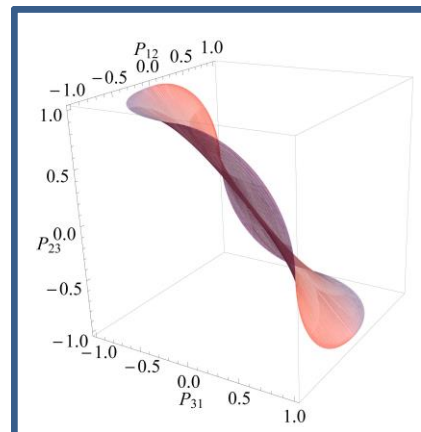

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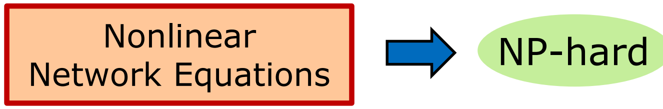

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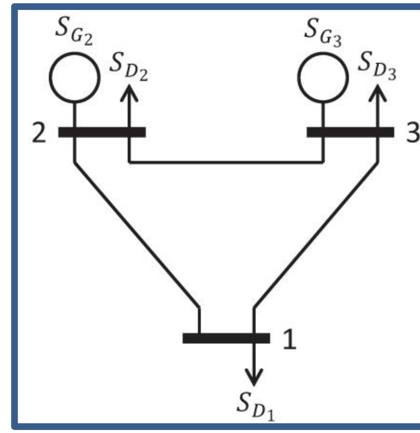

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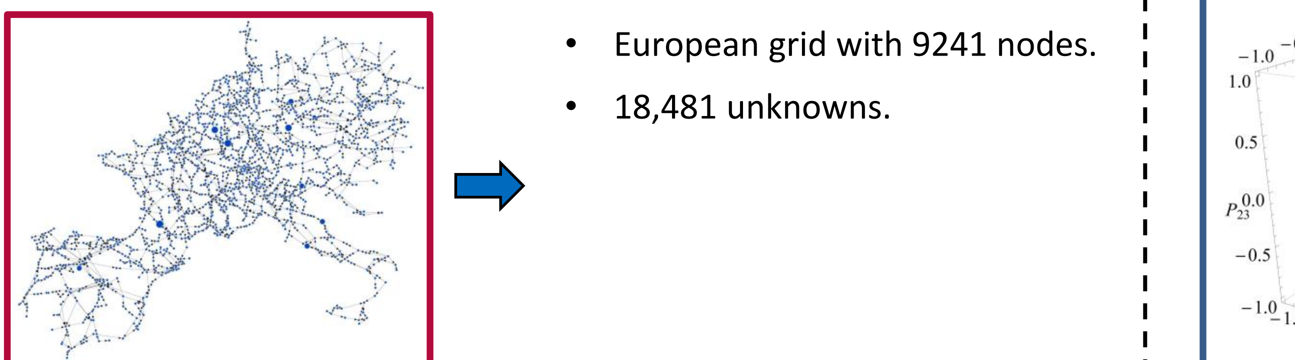
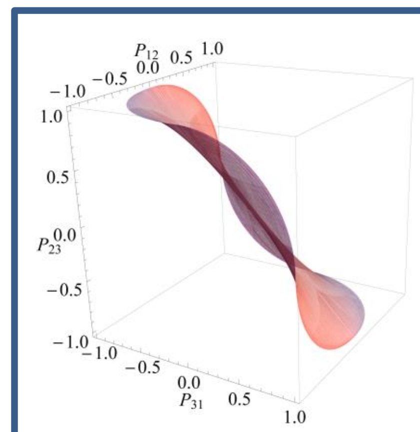


- European grid with 9241 nodes.
- 18,481 unknowns.

Feasible set:



Power System State Estimation (PSSE)

Noiseless Scenario: Power flow feasibility

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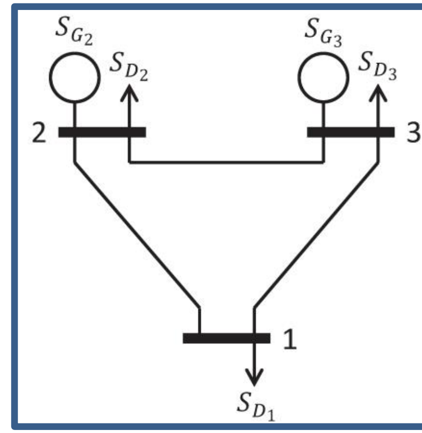

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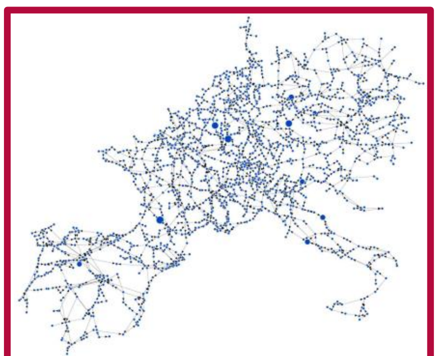
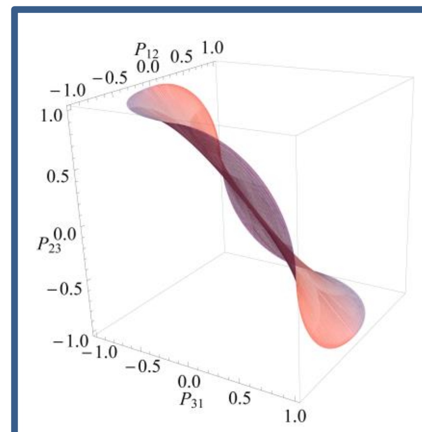


- European grid with 9241 nodes.
- 18,481 unknowns.
- 31,320 noisy measurements with 1% error.

Feasible set:



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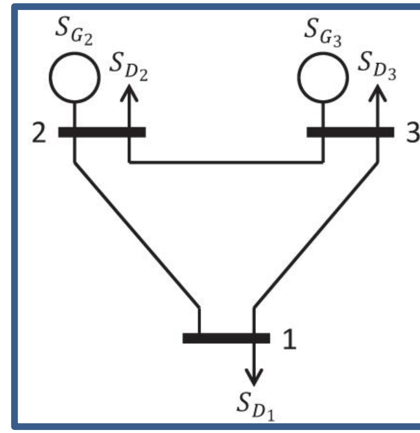

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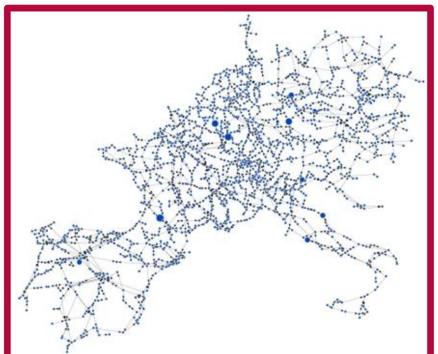

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Solution recovered with 0.5% accuracy

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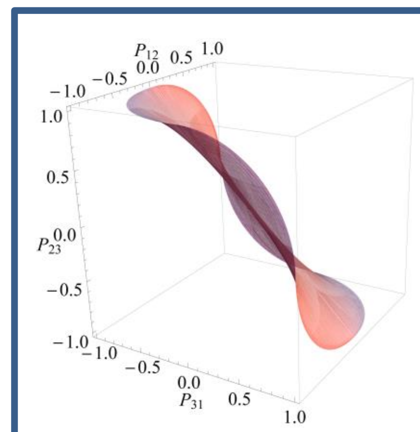

Quadratic Equations

- **State of the network:** $\mathbf{v} \in \mathbb{C}^n$

Noiseless Scenario:

$$\boxed{\left\{\begin{array}{rcl} \mathbf{v}^{\mathrm{H}}\,\mathbf{M}_1\,\mathbf{v} & = & y_1 \\ \mathbf{v}^{\mathrm{H}}\,\mathbf{M}_2\,\mathbf{v} & = & y_2 \\ & \vdots & \\ \mathbf{v}^{\mathrm{H}}\,\mathbf{M}_m\,\mathbf{v} & = & y_m \end{array}\right.}$$

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Vector of complex voltages

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Measured data

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Network model

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Noisy Case:

$$y_k = z_k + \omega_k + \eta_k$$

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Noisy Case:

$$\boxed{y_k = \underbrace{z_k}_{\text{True values}} + \omega_k + \eta_k}$$

Quadratic Equations

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Modest noise

Quadratic Equations

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Sparse bad data

Quadratic Equations

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$$m = O(n \log(n))$$

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  - Noise

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→ **Penalized semidefinite Programing (SDP)**

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Nonconvexity: Semidefinite Relaxation

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High dimensional space

$$\begin{cases} \langle \mathbf{M}_1, \mathbf{W} \rangle = y_1 \\ \langle \mathbf{M}_2, \mathbf{W} \rangle = y_2 \\ \quad\vdots \\ \langle \mathbf{M}_m, \mathbf{W} \rangle = y_m \\ \mathbf{W} \succeq 0. \end{cases}$$

Nonconvexity: Semidefinite Relaxation

- Lifting : $\boxed{v_i v_j^* \rightarrow W_{ij}}$

$$\left\{\begin{array}{rcl} \mathbf{v}^{\mathrm{H}}\,\mathbf{M}_1\,\mathbf{v} &=& y_1 \\ \mathbf{v}^{\mathrm{H}}\,\mathbf{M}_2\,\mathbf{v} &=& y_2 \\ &\vdots& \\ \mathbf{v}^{\mathrm{H}}\,\mathbf{M}_m\,\mathbf{v} &=& y_m \end{array}\right.$$

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Nonconvexity: Semidefinite Relaxation

- Lifting : $\boxed{v_i v_j^* \rightarrow W_{ij}}$
- The problem can be cast linearly w.r.t. to the matrix $\mathbf{W} \triangleq \mathbf{v}\mathbf{v}^{\mathrm{H}}$:

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- The matrix $\mathbf{W}$ is structured:

$$\boxed{\mathbf{W} \succeq 0,} \qquad \boxed{\mathrm{rank}\{\mathbf{W}\} = 1}$$

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Nonconvexity: Penalized SDP

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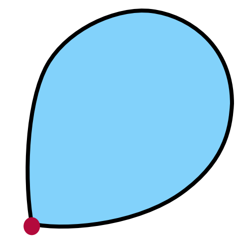
High dimensional set:



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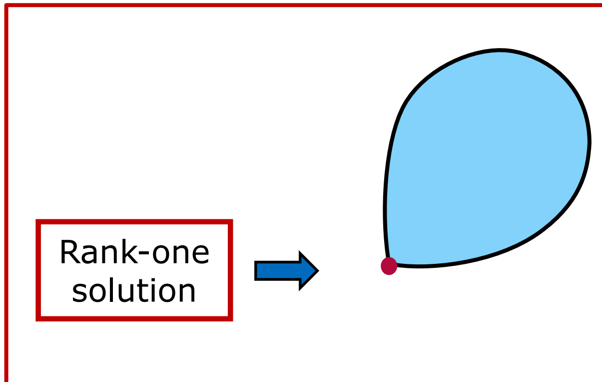

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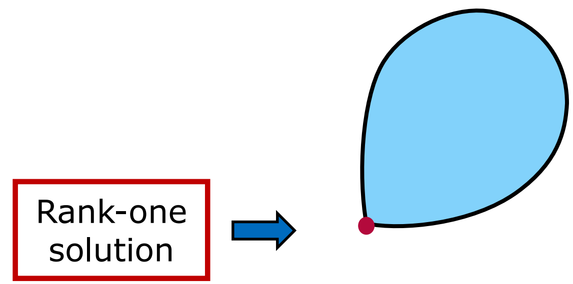

- Extra equations turn the feasible set into a single point.

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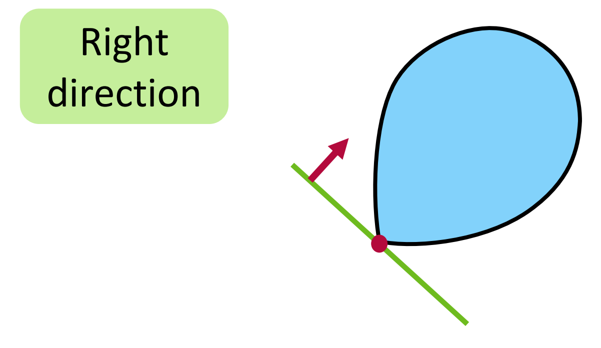

- Extra equations turn the feasible set into a single point.
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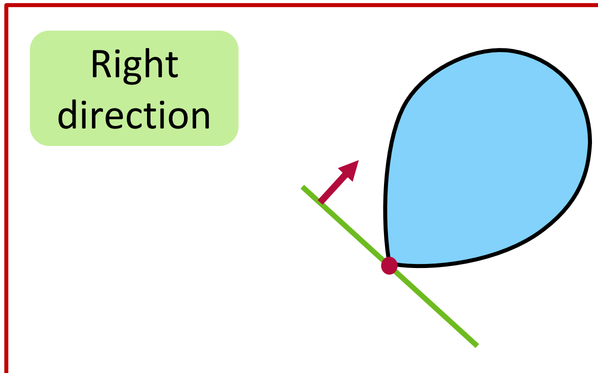

- Extra equations turn the feasible set into a single point.
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- Use $g(\mathbf{W})$ as the objective to build an optimization problem:

Penalized SDP:

$$\begin{aligned} \underset{\mathbf{W} \in \mathbb{H}_n}{\text{minimize}} \quad & g(\mathbf{W}) \\ \text{subject to} \quad & \langle \mathbf{M}_k, \mathbf{W} \rangle = y_k, \quad k = 1, \ldots, m \\ & \mathbf{W} \succeq 0 \end{aligned}$$

Nonconvexity: Penalty Design

How to design penalty functions for power systems applications?

$$\begin{array}{ll} \text{find} & \mathbf{v} \in \mathbb{C}^n \\ \text{subject to} & \mathbf{v}^{\mathrm{H}} \mathbf{M}_k \mathbf{v} = y_k, \;\; k = 1, \ldots, m \end{array}$$

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Theorem: Minimization of "reactive power" or "current magnitudes" both recover every solution in a regime where the phase angles are small.

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Guess for solution: $\hat{\mathbf{v}}$

1. $\mathbf{M} \succeq 0$
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3. 0 is a simple eigenvalue for $\mathbf{M}$

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Theorem: If there exists a solution in a vicinity of $\hat{\mathbf{v}}$ then SDP is exact.

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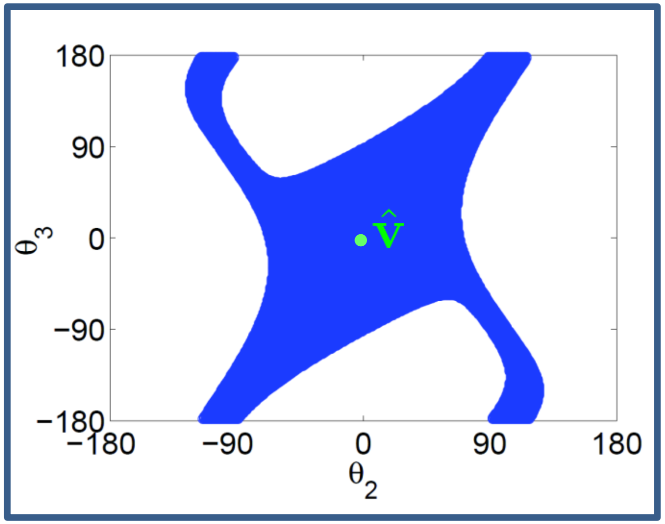
Guess for solution: $\hat{\mathbf{v}}$

1. $\mathbf{M} \succeq 0$
2. $\mathbf{M}\,\hat{\mathbf{v}} = 0$
3. 0 is a simple eigenvalue for $\mathbf{M}$

Theorem: If there exists a solution in a vicinity of $\hat{\mathbf{v}}$ then SDP is exact.

Recovery region $\mathcal{R}_{\mathbf{M}} \in \mathbb{C}^n$:



Nonconvexity: Penalty Design

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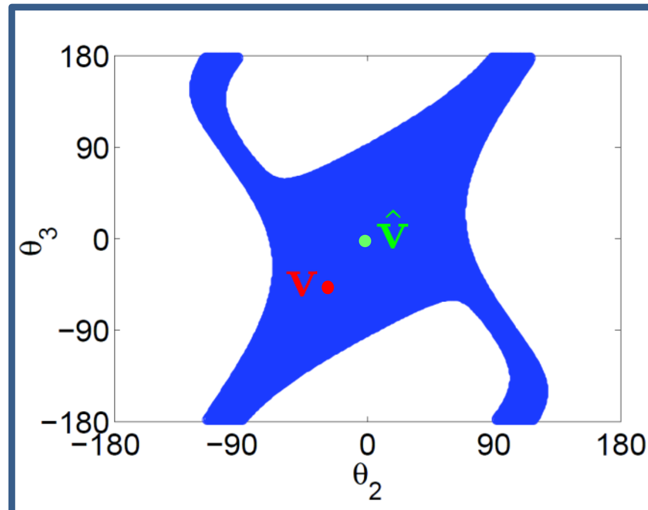

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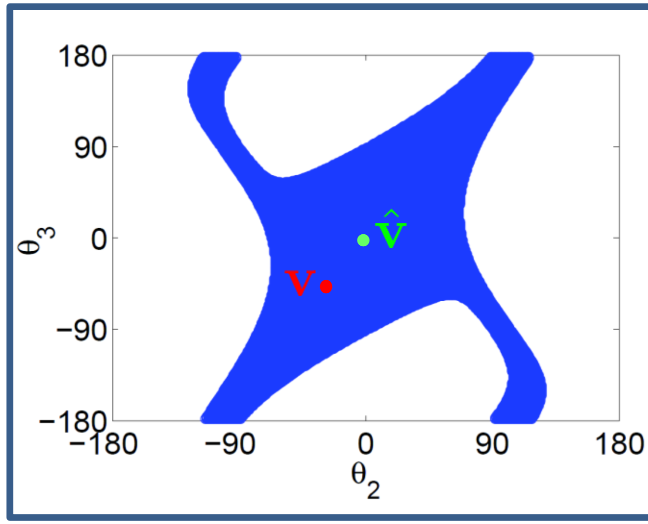
Theorem: If there exists a solution in a vicinity of $\hat{\mathbf{v}}$ then SDP is exact.

Theorem: There are finite number of $\mathbf{M}$'s that cover the entire space.

$\Rightarrow$

Recovery region $\mathcal{R}_{\mathbf{M}} \in \mathbb{C}^n$:



Comparisons with Newton's Method

Randomly chosen phase angles and voltage magnitude:

$$(\measuredangle v_1, \measuredangle v_2, \ldots, \measuredangle v_n) \in [-\theta, \theta]^n$$

$$(|v_1|, |v_2|, \ldots, |v_n|) \in [0.9, 1.1]^n$$

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➡

Derive the corresponding set of specifications

Recover an operating point using SDP

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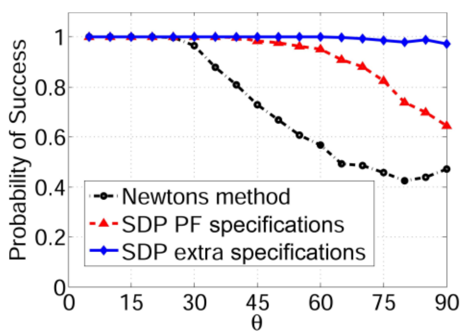
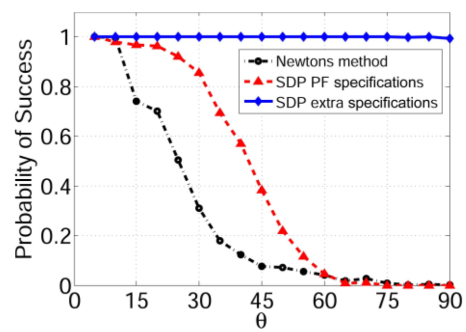
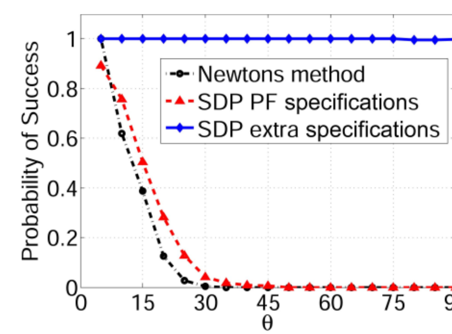


IEEE 9-bus system



New England 39-bus



IEEE 57-bus system

Comparisons with Newton's Method

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Derive the corresponding set of specifications

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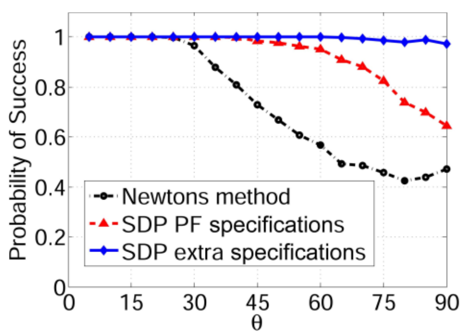
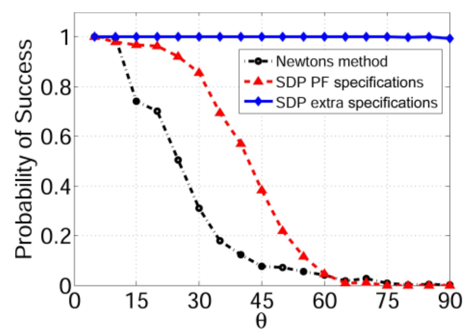


IEEE 9-bus system



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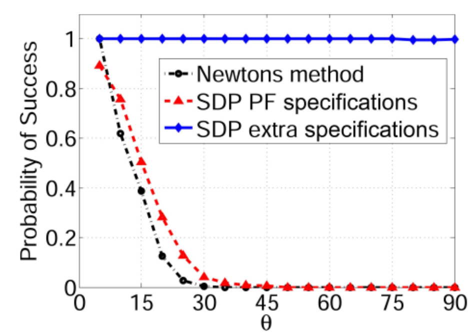

IEEE 57-bus system

- The proposed approach outperforms Newton’s method.

Comparisons with Newton's Method

Randomly chosen phase angles and voltage magnitude:

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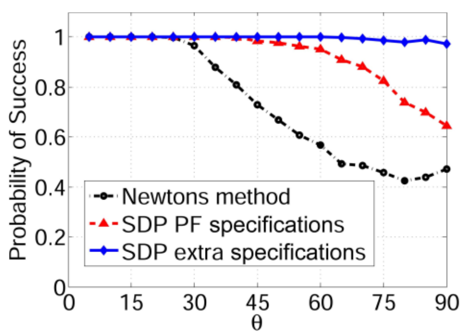
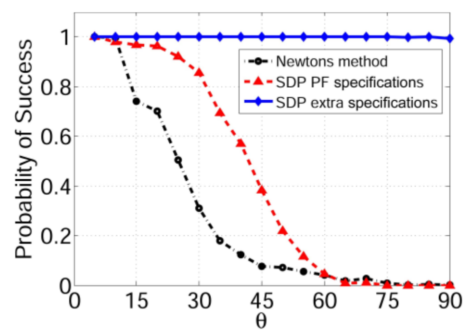
Derive the corresponding set of specifications

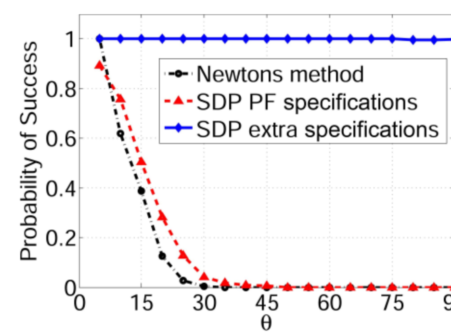
Recover an operating point using SDP



IEEE 9-bus system



New England 39-bus



IEEE 57-bus system

- The proposed approach outperforms Newton's method.
- Given additional equations, SDP is almost always exact.

Noisy Scenarios

How to handle noise?

$$\begin{array}{ll} \text{find} & \mathbf{v} \in \mathbb{C}^n \\ \text{subject to} & \mathbf{v}^{\mathrm{H}}\mathbf{M}_k\mathbf{v} + \omega_k = y_k, \;\; k = 1, \ldots, m \end{array}$$

$$\begin{array}{ll} \underset{\substack{\mathbf{W}\in\mathbb{H}_n \\ \boldsymbol{\nu}\in\mathbb{R}^m}}{\text{minimize}} & \langle \mathbf{M}, \mathbf{W} \rangle + \mu \times \|\boldsymbol{\nu}\|_1 \\ \text{subject to} & \langle \mathbf{M}_k, \mathbf{W} \rangle + \nu_k = y_k, \qquad k = 1, \ldots, m, \\ & \mathbf{W} \succeq 0 \end{array}$$

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Theorem (Noiseless Case): If Jacobian is nonsingular, the initial guess is sufficiently close and μ is large, then SDP+L1 is exact.

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Theorem (Small Noise): If $\mathbf{x}$ is recoverable with noiseless measurements, then every solution $(\mathbf{W}^{\text{opt}}, \boldsymbol{\nu}^{\text{opt}})$ of penalized SDP + L1 satisfies:

$$\|\mathbf{W}^{\text{opt}} - \alpha \mathbf{v}\mathbf{v}^*\|_F \leq c \times \sqrt{\mu \times \|\boldsymbol{\omega}\|_1 \times \text{trace}\{\mathbf{W}^{\text{opt}}\}}$$

Noisy Scenarios

How to handle noise?

$$\begin{aligned} \underset{\substack{\mathbf{W}\in\mathbb{H}_n \\ \boldsymbol{\nu}\in\mathbb{R}^m}}{\text{minimize}} \quad & \langle \mathbf{M}, \mathbf{W} \rangle + \mu \times \|\boldsymbol{\nu}\|_1 \\ \text{subject to} \quad & \langle \mathbf{M}_k, \mathbf{W} \rangle + \nu_k = y_k, \qquad k = 1, \ldots, m, \\ & \mathbf{W} \succeq 0 \end{aligned}$$

Elevates the likelihood of estimated noise values

1- Y. Zhang, R. Madani and J. Lavaei, "Power System State Estimation with Line Measurements," Preprint, 2016.

Noisy Scenarios

How to handle noise?

$$
\begin{aligned}
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Elevates the likelihood of estimated noise values

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- What is the statistical behavior of the bound?
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Elevates the likelihood of estimated noise values

- What is the statistical behavior of the bound?
- How does the number of measurements effect the quality of estimation?
- Define the root-mean-square error:

$$\zeta := \frac{\|\mathbf{W}^{\text{opt}} - \alpha \mathbf{v}\mathbf{v}^*\|_F}{\sqrt{n \times \text{trace}\{\mathbf{W}^{\text{opt}}\}}}$$

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$\Rightarrow$ Elevates the likelihood of estimated noise values

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- How does the number of measurements effect the quality of estimation?
- Define the root-mean-square error:

$$\zeta := \frac{\|\mathbf{W}^{\text{opt}} - \alpha \mathbf{v}\mathbf{v}^*\|_F}{\sqrt{n \times \text{trace}\{\mathbf{W}^{\text{opt}}\}}}$$

Theorem: Under recoverability assumption, the tail probability of the root-mean-square estimation error is upper bounded as:

$$\mathbb{P}(\zeta > t) \leq \mathrm{e}^{-\gamma(t)m} \quad \text{where} \quad \gamma(t) := \frac{t^4 \lambda^2}{32\kappa^2\rho^2} - \ln 2.$$

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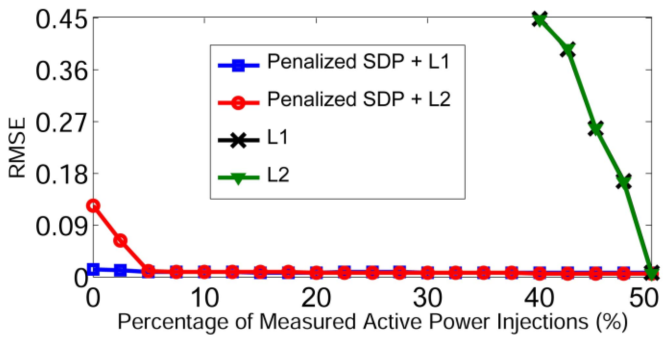
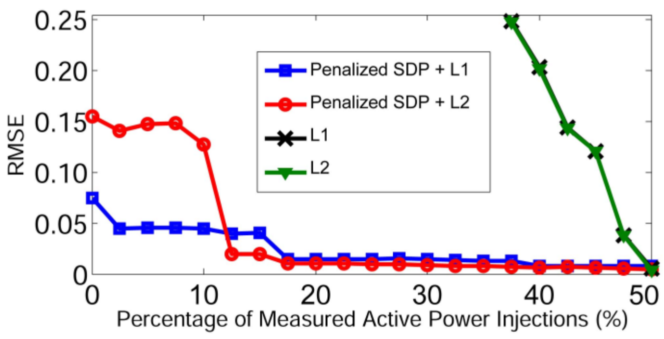
Case Studies

- A 1354 node European grid with 1% and 2% measurement noise values:





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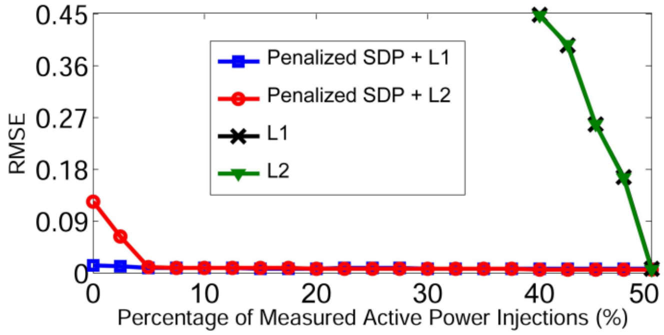
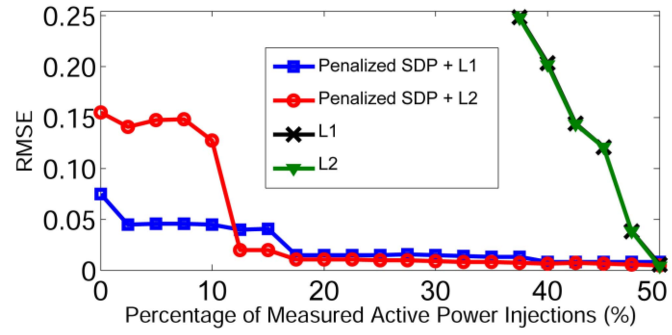
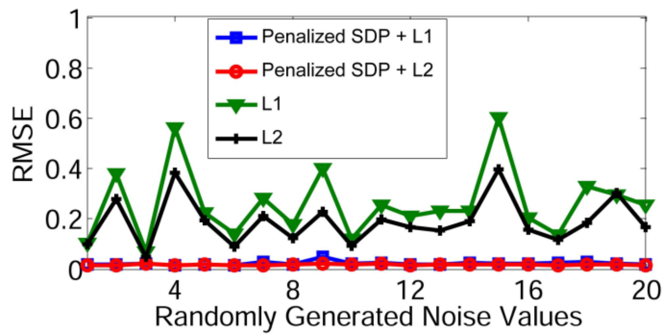


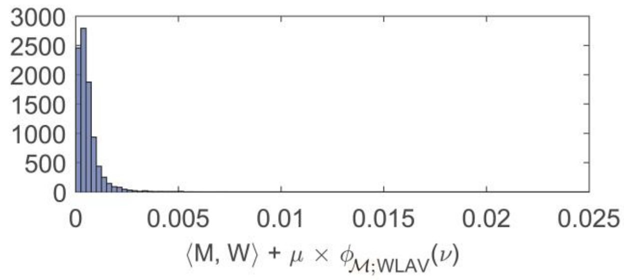


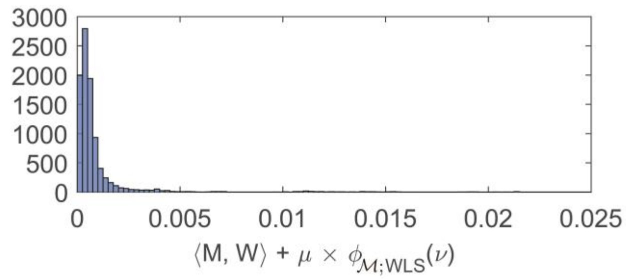
- Random realizations with 5% measurement noise and 30% extra equations:



Case Studies

- Histograms of absolute differences between the actual and estimated complex voltages for the PEGASE 9241-bus system, using the penalized convex problem equipped with the WLAV and WLS estimators respectively:





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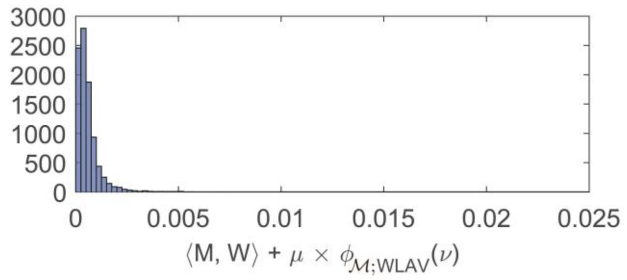
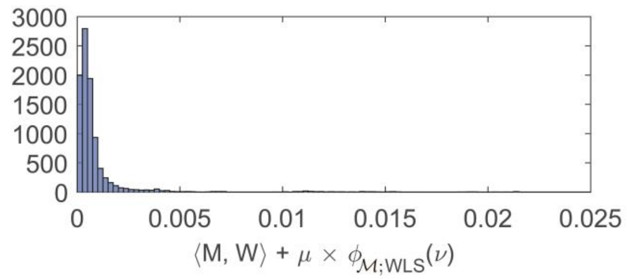
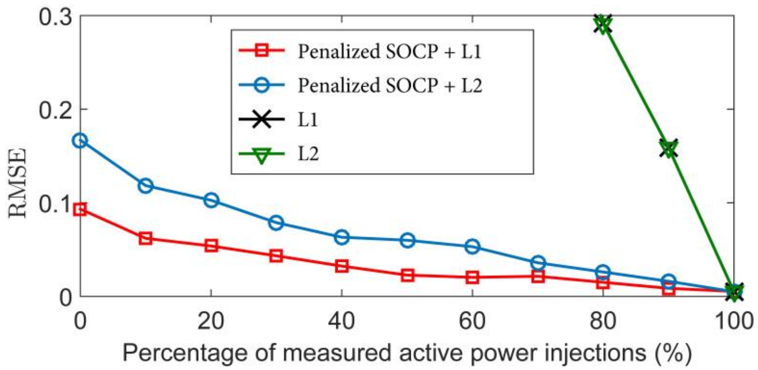


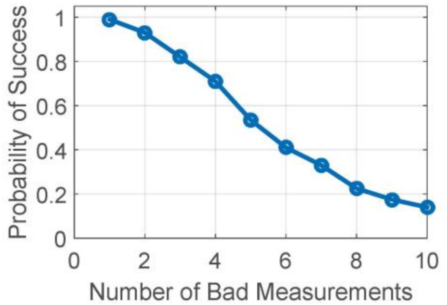


- Average RMSE of the estimated voltages obtained by SOCP over 10 Monte-Carlo simulations for different noise realizations:

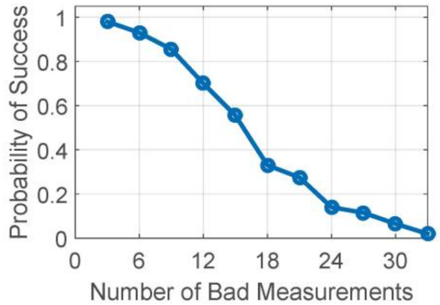


Case Studies

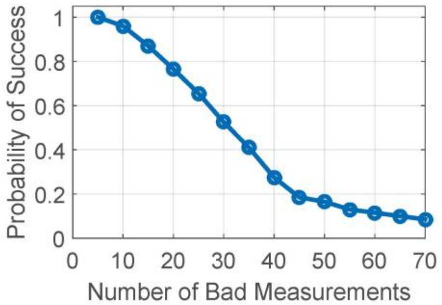
- The probability of success for the penalized SDP problem when different numbers of measurements are corrupted for New England 39-bus, IEEE 57-bus and IEEE 118-bus systems, respectively:



New England 39-bus

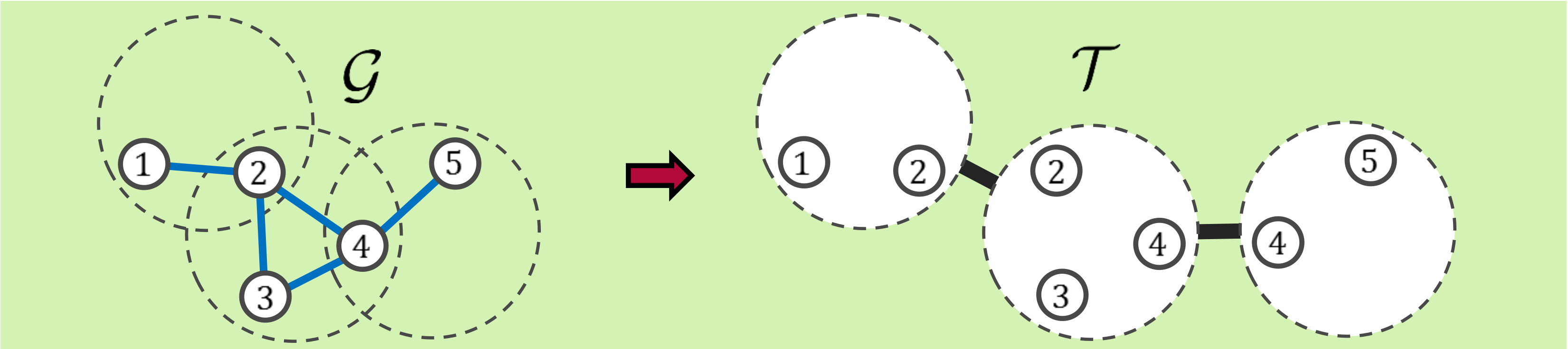


IEEE 57-bus

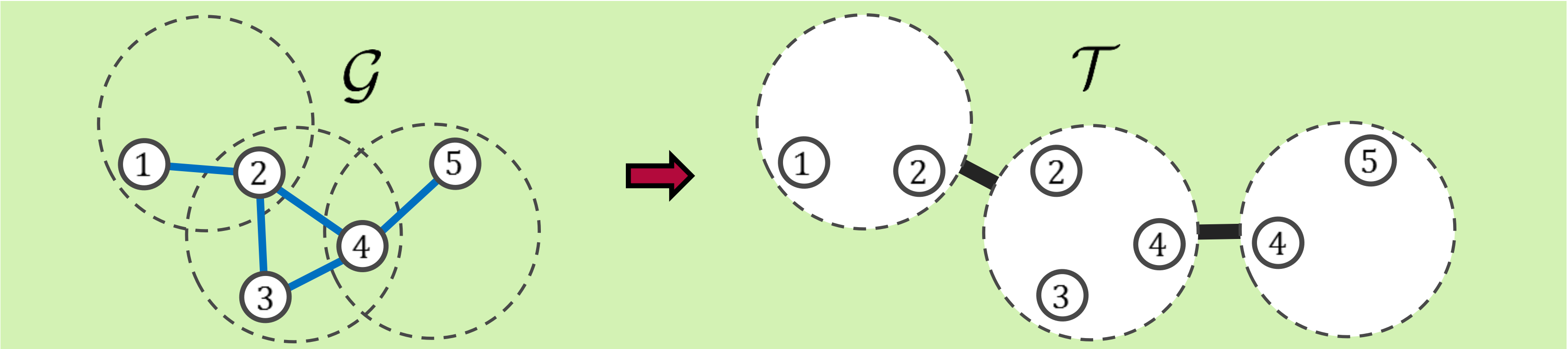


IEEE 118-bus

Tree Decomposition

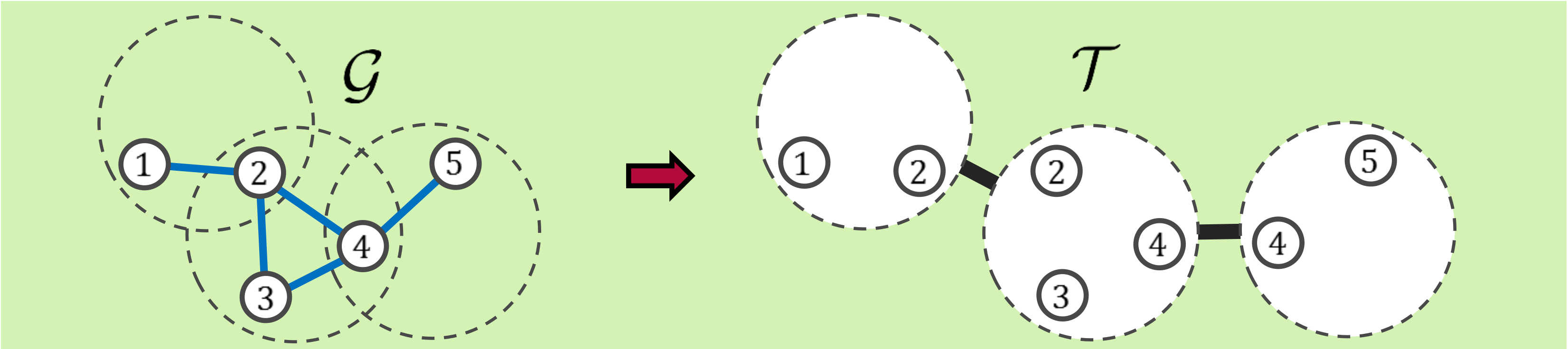


Tree Decomposition



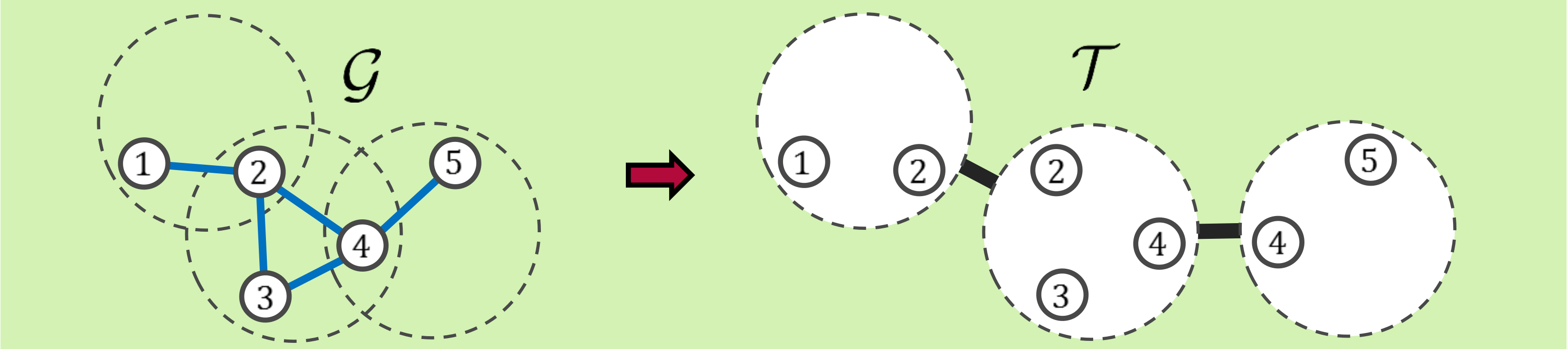
- **Tree decomposition:** A way of grouping the vertices in order to turn the graph into a tree [Robertson and Seymour, 84].

Tree Decomposition



- **Tree decomposition:** A way of grouping the vertices in order to turn the graph into a tree [Robertson and Seymour, 84].
- **Tree width:** The most efficient way of grouping in terms of the size for largest group.

Tree Decomposition



- **Tree decomposition:** A way of grouping the vertices in order to turn the graph into a tree [Robertson and Seymour, 84].
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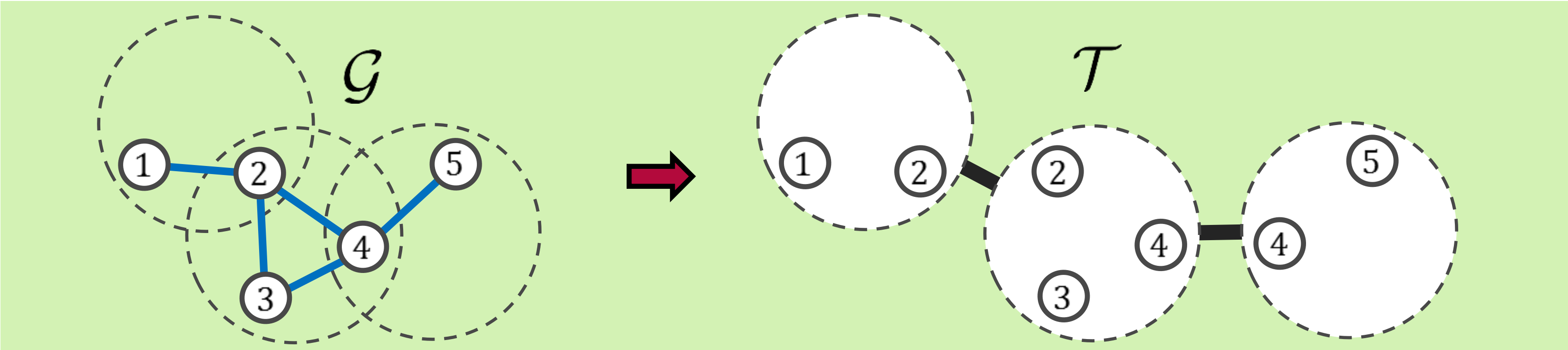
- Very efficient methods to calculating upper bounds [Bodlaender, 11].

Tree Decomposition



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TW of the New York power grid with 8500 nodes ≤ 40.

TW of the European power grid 9000 nodes ≤ 31.

Decomposed SDP

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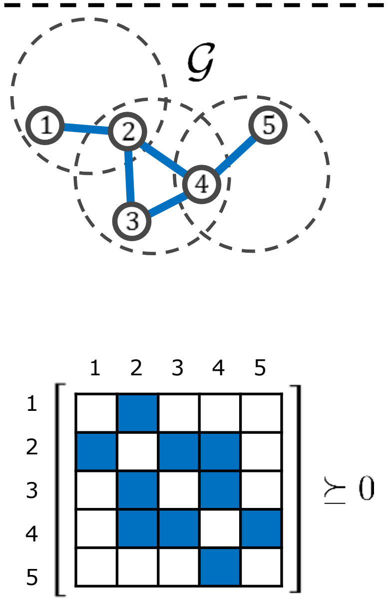

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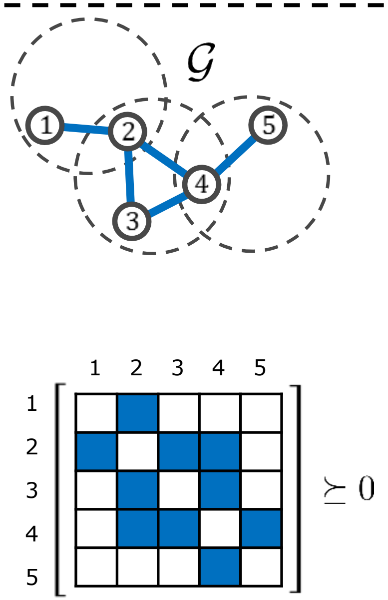

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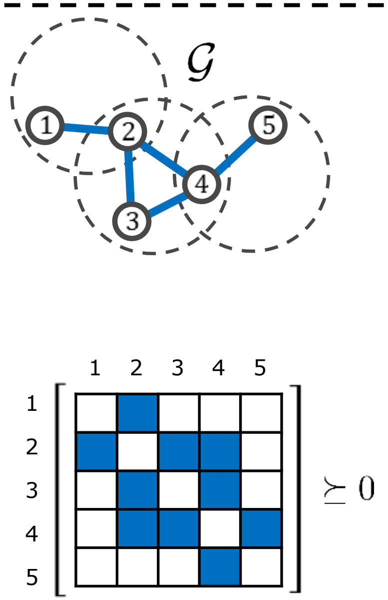

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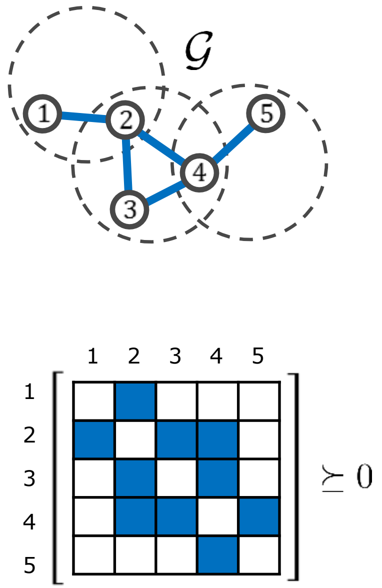

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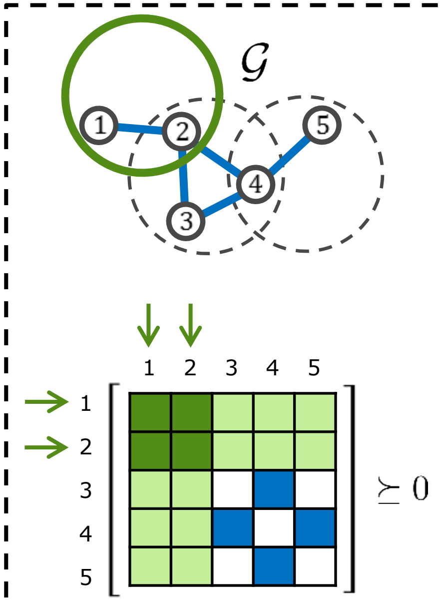

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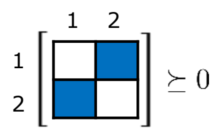

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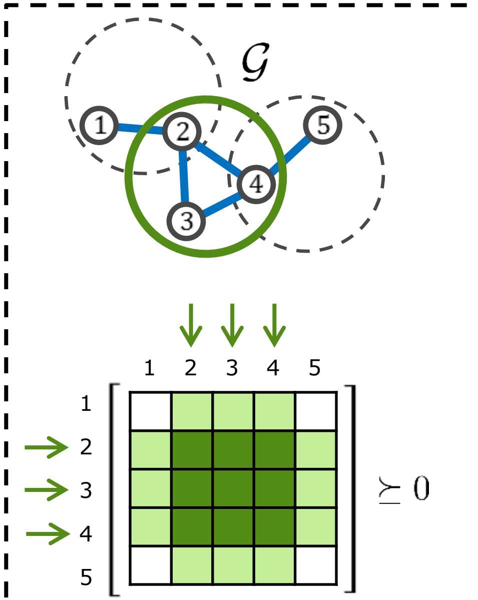

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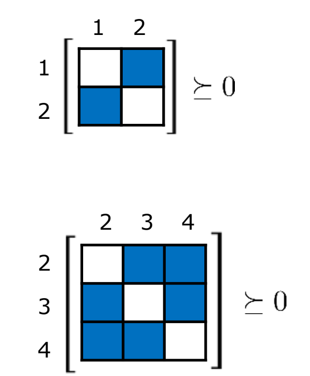

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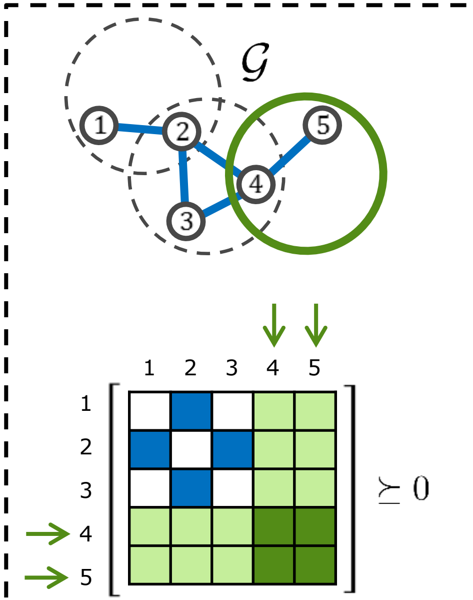

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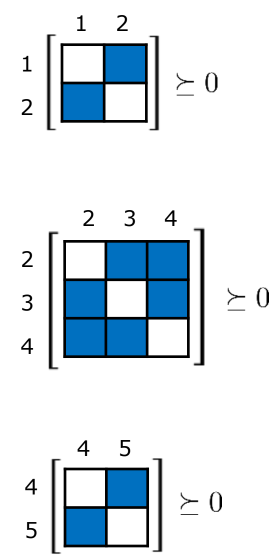

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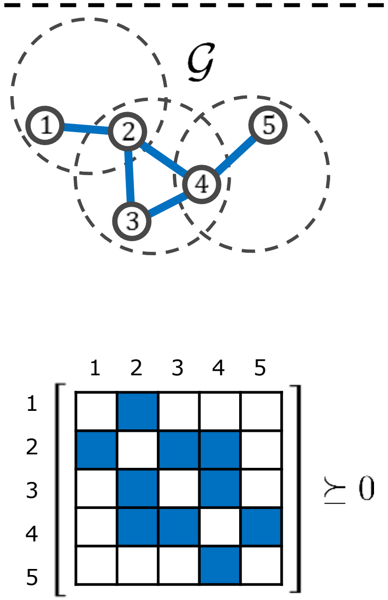

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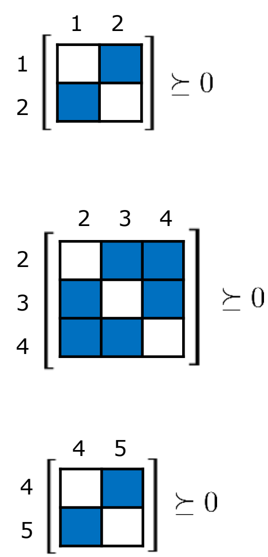

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Conclusions

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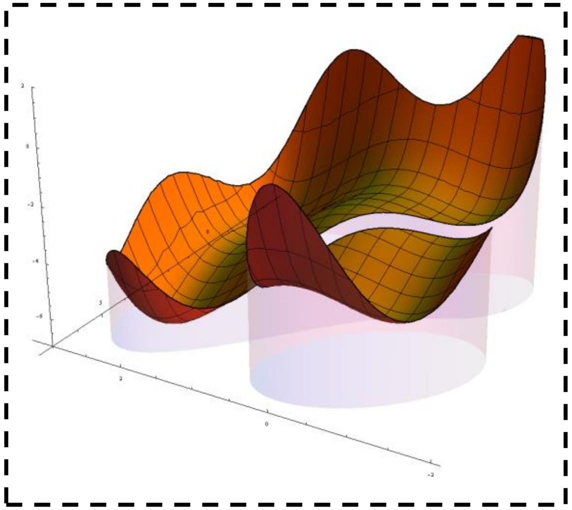
  - Polynomial Optimization
  - Polynomial Feasibility
- Advantages:
  - Guaranteed rank one solution
  - Rejection of bad data
  - Robustness to noise
- Applications:
  - Optimal Power Flow
  - Power System State Estimation
  - Tensor completion



Thank you