Valid Inequalities for the Alternating Current Optimal Power Flow (ACOPF)

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Overview

Introduction to ACOPF & formulations

Conic relaxations and sparsity

- Valid inequalities for a semidefinite program
- Valid inequalities in the case of duality gap

Future research and conclusion

ACOPF, in brief

Minimize single-period 'costs' of operating a power system under static (steady-state) conditions

'Costs': Real-power generation (economic dispatch), losses, control-shifts,…

Things to model: basic network elements, transformer taps, FACTS, phase shifters…

Motivation

We explore the question:

How can one improve a lower bound for ACOPF?

Second-order cone (SOCP) relaxation with mesh gap

Semidefinite (SDP) relaxation with duality gap

Note: We use real power cost minimization and take as constraints only the bare bones of ACOPF

Formulating ACOPF

Polar (POL)

Variables: $P, Q, |V|, \theta$

Variable bounds: $P^{\min}, P^{\max}, Q^{\min}, Q^{\max}, \theta_{in}^{\max}$

Admittance: $Y = G + jB$

$\min \sum_i c_1 P_i^2 + c_2 P_i$

s.t.

Power Flow Equations

$$P_i = \sum_{n \in N} [G_{in}|V_i||V_n|\cos(\theta_{in}) + B_{in}|V_i||V_n|\sin(\theta_{in})], \forall i \in N$$

$$Q_i = \sum_{n \in N} [G_{in}|V_i||V_n|\sin(\theta_{in}) - B_{in}|V_i||V_n|\cos(\theta_{in})], \forall i \in N$$

Variable Bounds

$$P^{\min} \leq P \leq P^{\max}$$

$$Q^{\min} \leq Q \leq Q^{\max}$$

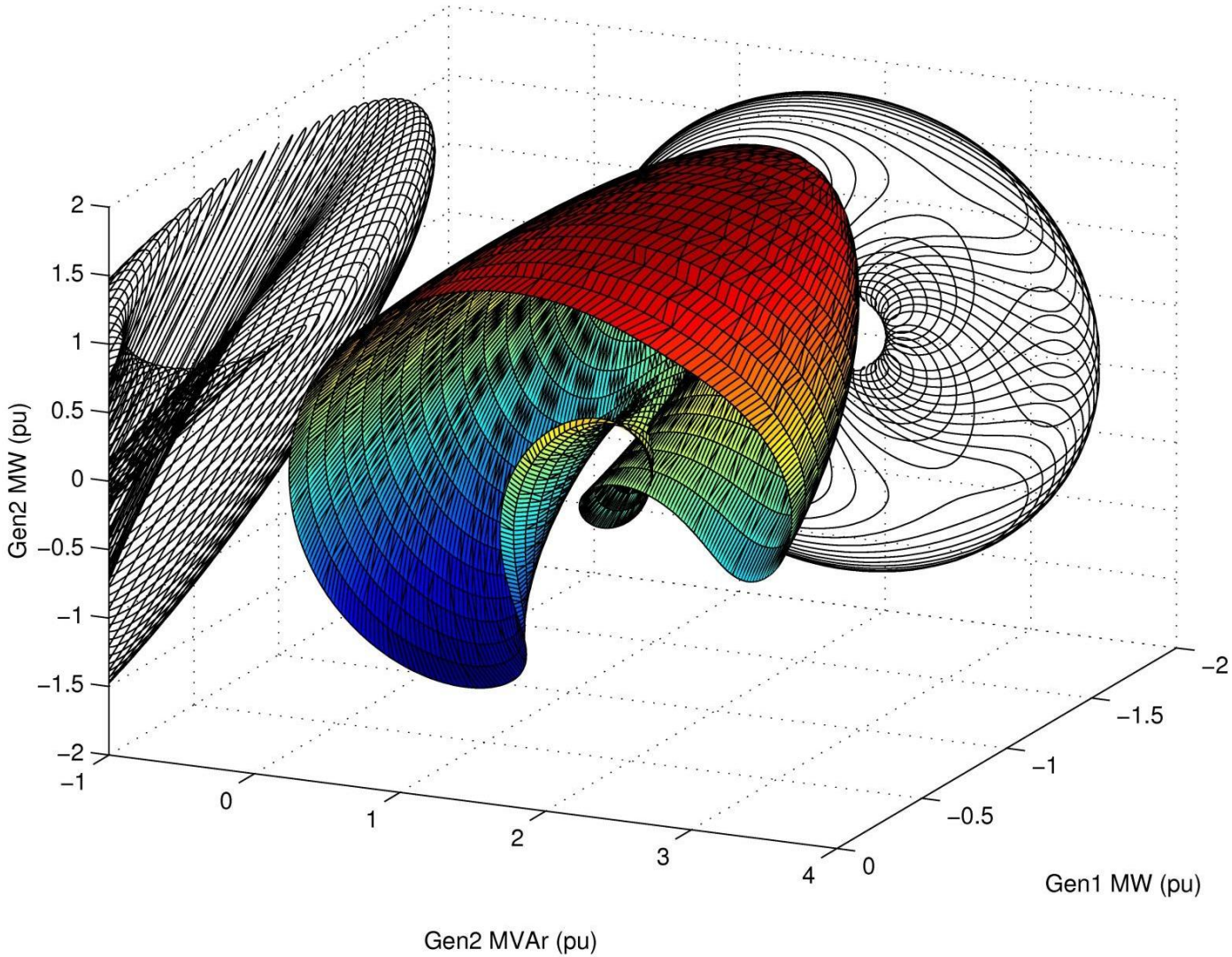
$$V^{\min} \leq |V| \leq V^{\max}$$

$$-\theta_{in}^{\max} \leq \theta_i - \theta_n \leq \theta_{in}^{\max}, \forall i, n \in E$$

Formulating ACOPF

Nooks and crannies

I.A. Hiskens and R.J. Davy, "Exploring the power flow solution space boundary", *IEEE Transactions on Power Systems*, Vol. 16, No. 3, August 2001, pp. 389-395.



Formulating ACOPF

Rectangular (REC)

$V := V^{Re} + jV^{Im}, V^{Re} := |V|\cos(\theta) \geq 0$, and $V^{Im} := |V|\sin(\theta)$

$V^* = (V^{Re})^T - j(V^{Im})^T$

$\min \sum_i c_1 P_i^2 + c_2 P_i$

s.t.

$$P + jQ = diag(YVV^*)$$

$$P^{\min} \leq P \leq P^{\max}$$

$$Q^{\min} \leq Q \leq Q^{\max}$$

$$V^{\min^2} \leq \mathrm{diag}(VV^*) \leq V^{\max^2}$$

$$\pm \mathrm{imag}(V_i V_n^*) \leq \mathrm{real}(V_i V_n^*) \tan(\theta_{in}^{\max}), \ \forall i, n \in E$$

Formulating ACOPF

REC as QCQP

$$V^{Re} \in \mathbb{R}_{+}^{|N|}, V^{Im} \in \mathbb{R}^{|N|}$$

$$\min \sum_i c_1 P_i^2 + c_2 P_i$$
$$\text{s.t.}$$

$$P_i = \sum_{n \in N} [G_{in}(V_i^{Re} V_n^{Re} + V_i^{Im} V_n^{Im}) - B_{in}(V_i^{Re} V_n^{Im} - V_i^{Im} V_n^{Re})], \ \forall i \in N$$

$$Q_i = -\sum_{n \in N} [B_{in}(V_i^{Re} V_n^{Re} + V_i^{Im} V_n^{Im}) + G_{in}(V_i^{Re} V_n^{Im} - V_i^{Im} V_n^{Re})], \ \forall i \in N$$

$$P^{\min} \leq P \leq P^{\max}$$

$$Q^{\min} \leq Q \leq Q^{\max}$$

$$(V_i^{\min})^2 \leq (V_i^{Re})^2 + (V_i^{Im})^2 \leq (V_i^{\max})^2, \ \forall i \in N$$

$$\pm (V_i^{\mathrm{Re}} V_n^{\mathrm{Im}} - V_i^{\mathrm{Im}} V_n^{\mathrm{Re}}) \leq (V_i^{\mathrm{Re}} V_n^{\mathrm{Re}} + V_i^{\mathrm{Im}} V_n^{\mathrm{Im}}) \tan(\theta_{in}^{\max}), \ \forall i, n \in E$$

Overview of Relaxations

A relaxation is a special approximation that helps global optimality

- Contains original feasible region (WLOG assume linear objective)
- Lower (upper) bound on optimal min (max)
  - Proof of infeasibility
  - Feasible -> Optimal

'Easy' relaxations

- Self-dual convex cones (conic programming)
  - Nonnegative Orthant (LP)
  - Second-Order (SOCP)
  - Positive Semidefinite (SDP)
- Convex Quadratically Constrained Quadratic Prog. (QCQP), Geometric Programming, etc.

Relaxations

Nonconvex QCQPs

$$\begin{aligned} \min \;\; & x^T A_0 x + a_0^T x + b_0 \\ s.t. \; & x^T A_i x + a_i^T x + b_i \leq 0, i = 1, ..., m \end{aligned}$$

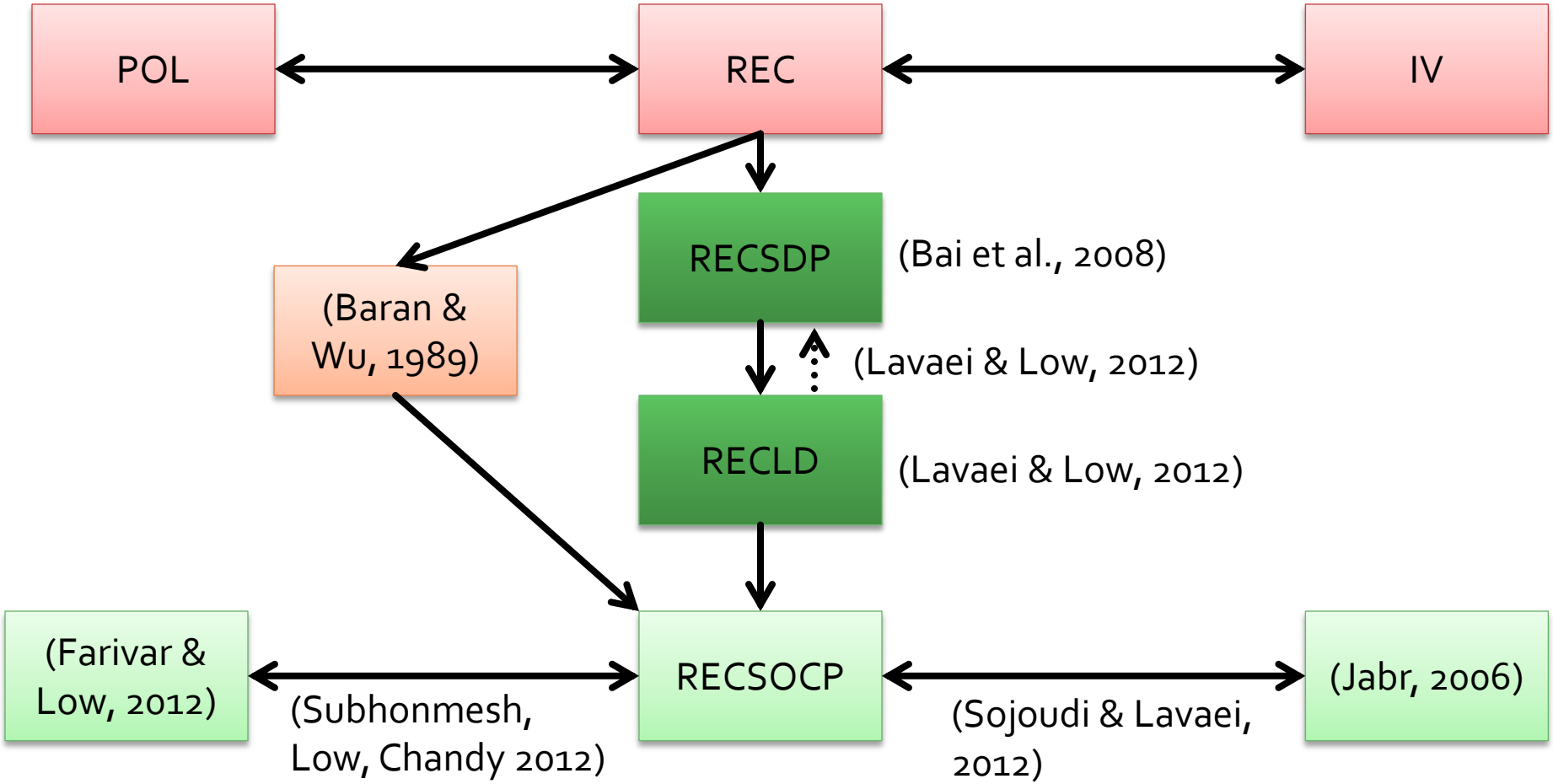
Can model: binary variables, complementarity constraints, conic programming, etc.

Standard tools for generating relaxations

- Linear Programming
  - Outer-Approximation (McCormick , 1982) ,(Al Khayyal & Falk 1983)
  - Reformulation-Linearization Technique (RLT) (Sherali & Tuncbilek, 1992)
- Semidefinite Programming
  - Lagrangian Dual via Linear Matrix Inequalities (Shor, 1987)
  - Hierarchy of SDP relaxations (Lasserre, 2001)
- Second-Order Cone Programming
  - Relaxations of a positive semidefinite (PSD) constraint (Kim & Kojima, 2001), (Muramatsu & Suzuki, 2003)

Relaxations

ACOPF Map



(Sojoudi & Lavaei, 2012)
(Bose et al., 2012)
(Zhang & Tse, 2011)

Relaxations

Lifting REC

$W + jT := VV^*$

$W + jT$ is a Hermitian, so W is symmetric and T skew-symmetric.

$\min \sum_i c_1 P_i^2 + c_2 P_i$

s.t.

$$P + jQ = diag(Y(W + jT))$$

$$P^{\min} \leq P \leq P^{\max}$$

$$Q^{\min} \leq Q \leq Q^{\max}$$

$$V^{\min^2} \leq \mathrm{diag}(W) \leq V^{\max^2}$$

$$\pm T_{in} \leq W_{in} \tan(\theta_{in}^{\max}), \ \forall i, n \in E$$

$$W \geq 0$$

$$W_{in} = V_i^{\mathrm{Re}} V_n^{\mathrm{Re}} + V_i^{\mathrm{Im}} V_n^{\mathrm{Im}}, \ \forall i, n$$

$$T_{in} = V_i^{\mathrm{Re}} V_n^{\mathrm{Im}} - V_i^{\mathrm{Im}} V_n^{\mathrm{Re}}, \ \forall i, n$$

Removing REC variables

$$W + jT = VV^*$$
$$\equiv W + jT \succeq 0, \quad rank(W + jT) = 1$$
$$\equiv \begin{bmatrix} W & T \\ -T & W \end{bmatrix} \succeq 0, \quad W_{ij}^2 + T_{ij}^2 >= W_{ii}W_{jj}$$

Relaxations

RECSDP (complex variables)

$\min \sum_i c_1 P_i^2 + c_2 P_i$

s.t.

$$P + jQ = diag(Y(W + jT))$$

$$P^{\min} \leq P \leq P^{\max}$$

$$Q^{\min} \leq Q \leq Q^{\max}$$

$$V^{\min^2} \leq \mathrm{diag}(W) \leq V^{\max^2}$$

$$\pm T_{in} \leq W_{in} \tan(\theta_{in}^{\max}), \ \forall i, n \in E$$

$$W \geq 0$$

$$W + jT \succeq 0$$

Relaxations

RECSDP

$$\min \sum_i c_1 P_i^2 + c_2 P_i$$

s.t.

$$P = G_{in} W_{in} - B_{in} T_{in}, \ \forall i, n \in E$$

$$Q = B_{in} W_{in} + G_{in} T_{in}, \ \forall i, n \in E$$

$$P^{\min} \leq P \leq P^{\max}$$

$$Q^{\min} \leq Q \leq Q^{\max}$$

$$V^{\min^2} \leq \mathrm{diag}(W) \leq V^{\max^2}$$

$$\pm T_{in} \leq W_{in} \tan(\theta_{in}^{\max}), \ \forall i, n \in E$$

$$W \geq 0$$

$$\begin{bmatrix} W & T \\ -T & W \end{bmatrix} \succeq 0$$

Relaxations

Lagrangian Dual as SDP (RECLD)

$$\max \sum_{n\in\mathcal{N}} [\underline{\lambda}_n(P_n^{\min} - P_n^d) - \overline{\lambda}_n(P_n^{\max} - P_n^d)$$
$$+ \underline{\gamma}_n(Q_n^{\min} - Q_n^d) - \overline{\gamma}_n(Q_n^{\max} - Q_n^d) \quad + \underline{\mu}_n(V_n^{\min})^2 - \overline{\mu}_n(V_n^{\max})^2]$$
$$+ \sum_{g\in\mathcal{G}} (c_{g0} + c_{g1} + 2\sqrt{c_{g2}} r_g - r_g^2)$$

subject to

$$\sum_{n\in\mathcal{N}} [Y_n^P(\overline{\lambda}_n - \underline{\lambda}_n) + Y_n^Q(\overline{\gamma}_n - \underline{\gamma}_n) + M_n\mu_n] + \sum_{g\in\mathcal{G}} [c_{g1} + 2\sqrt{c_{g2}} r_g] \succeq 0$$

$$\underline{\lambda}_n, \overline{\lambda}_n, \underline{\gamma}_n, \overline{\gamma}_n, \underline{\mu}_n, \overline{\mu}_n \geq 0$$

Relaxations

RECSOCP (complex variables)

$$W + jT \succeq 0 \implies \begin{vmatrix} W_{ii} & W_{in} + jT_{in} \\ W_{in} - jT_{in} & W_{nn} \end{vmatrix} \geq 0$$

$$\equiv W_{ii}W_{nn} \geq |W_{in} + jT_{in}|^2, \;\; W_{ii}, W_{nn} \geq 0$$

$$\equiv W_{ii} + W_{nn} \geq ||(2W_{in}, 2T_{in}, W_{ii} - W_{nn})||_2$$

$$\min \textstyle\sum_i c_1 P_i^2 + c_2 P_i$$

s.t.

$$P + jQ = diag(Y(W + jT))$$

$$P^{\min} \leq P \leq P^{\max}$$

$$Q^{\min} \leq Q \leq Q^{\max}$$

$$V^{\min^2} \leq \mathrm{diag}(W) \leq V^{\max^2}$$

$$\pm T_{in} \leq W_{in} \tan(\theta_{in}^{\max}), \;\; \forall i, n \in E$$

$$W \geq 0$$

$$W_{ii}W_{nn} \geq |W_{in} + jT_{in}|^2, \;\; W_{ii}, W_{nn} \geq 0$$

Relaxations

RECSOCP

$$\min \sum_i c_1 P_i^2 + c_2 P_i$$

s.t.

$$P = G_{in} W_{in} - B_{in} T_{in}, \ \forall i, n \in E$$

$$Q = B_{in} W_{in} + G_{in} T_{in}, \ \forall i, n \in E$$

$$P^{\min} \leq P \leq P^{\max}$$

$$Q^{\min} \leq Q \leq Q^{\max}$$

$$V^{\min^2} \leq \mathrm{diag}(W) \leq V^{\max^2}$$

$$\pm T_{in} \leq W_{in} \tan(\theta_{in}^{\max}), \ \forall i, n \in E$$

$$W \geq 0$$

$$W_{ii} + W_{nn} \geq ||(2W_{in}, 2T_{in}, W_{ii} - W_{nn})||_2$$

Exploiting Sparsity

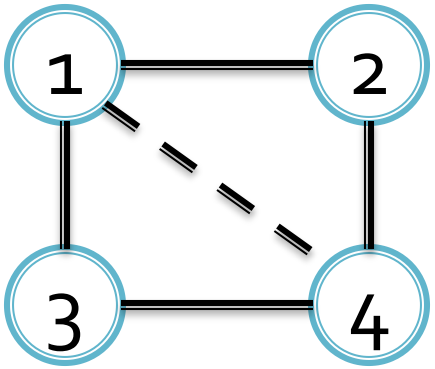
(Fukuda et al., 2000): from the completion theorem of (Grone et al., 1984), we can decompose one PSD constraint into PSD constraints on principal submatrices corresponding to *cliques of a chordal extension graph*.

- Reduces number of variables
  - Finding a minimal extension is NP-Complete
  - An 'optimal' extension is not always minimal
- Can be used to prove RECSOCP = RECSDP for tree networks
- Finding minimal completions is NP-complete

ACOPF applications:
(Bai & Wei, 2011): Hierarchy heuristic
(Sojoudi & Lavaei, 2012): Cycle basis
(Jabr, 2012): Cholesky decomposition heuristic
(Molzahn et al., 2013): Custom greedy heuristic

Chordal completion

A chordal completion is a set of additional edges that will make a graph chordal.



It's sufficient to consider the cliques {1,2,4} and {1,3,4}. For ACOPF, we could replace $W + jT \succeq 0$ with two PSD constraints and discard the lifting variables W_{23}, T_{23}

Valid Inequalities & Cuts

Sparse RECSDP

Matrix Variables Usage (Cholesky heuristic)						
Buses	Connected bus pairs	Additional pairs	**% of possible variables**	# Cliques	Avg Clique Size	Max Clique Size
14	20	3	**36%**	12	2.9	3
30	41	4	**18%**	26	3	4
39	46	21	**14%**	34	2.8	4
57	78	59	**12%**	52	3.4	6
118	179	84	**5%**	109	3.3	5
300	409	255	**2%**	249	3.2	8

Numerical comparisons

Hardware: 2.26 GHz dual-core Intel, 4GB RAM

Language: MATLAB r2010a

Modeling: YALMIP (05-April-2013)

Optimization: MOSEK 7.0.0.46, IPOPT

Data: MATPOWER 4.1

Relaxations

Time to solve (s)

Buses	RECSDP	Sparse RECSDP	RECLD	RECSOCP	Sparse RECLD (Molzahn et al., 2013)
14	0.9	0.1	2.5	0.1	-
30	12.6	0.2	0.8	0.1	-
39	68.8	0.3	1.9	0.1	-
57	**525.8**	0.5	2.5	0.1	-
118	-	**1**	18.2	0.2	2.1
300	-	- (memory)	363.1	0.6	5.7
2383	-	-	-	7.3	730

Quality of RECSOCP Bound

Zero Duality Gap

Buses	% of best primal solution (IPOPT)
14	99.9%
30	99.8%
39	99.98%
57	99.9%
118	99.8%
300	99.9%
2383	99.3%

Exactness

Literature has sought conditions where RECSDP/RECSOCP solves REC.

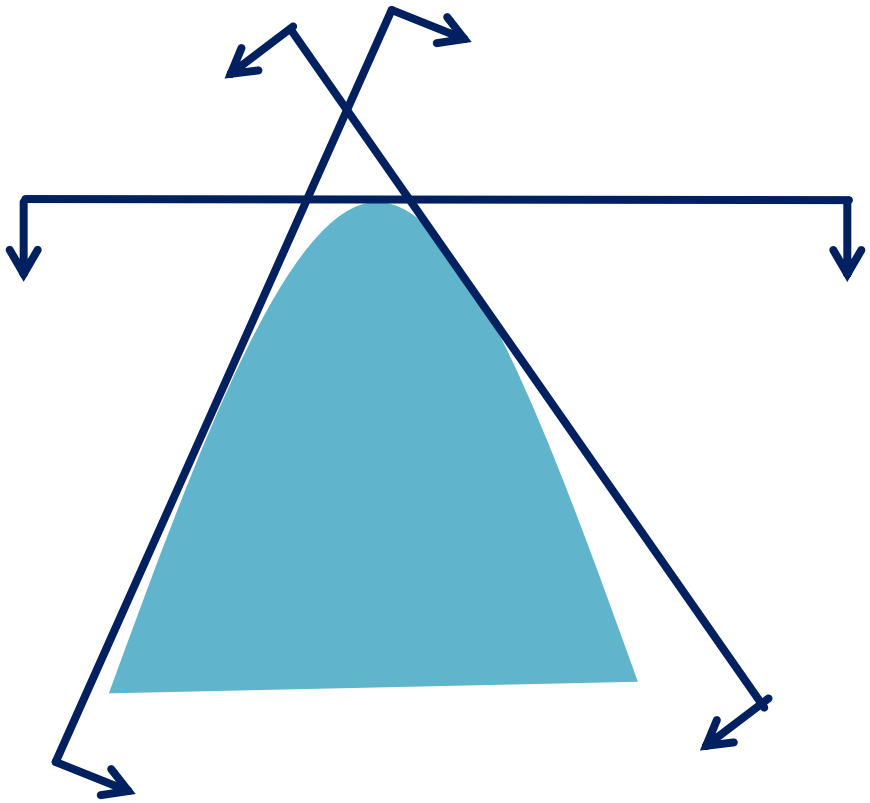
We focus on the **gaps**:

- 'Mesh gaps' for RECSOCP
- Duality gaps for RECSDP

Valid Inequalities & Cuts (Mesh Gaps)

Outer Approximation of PSD cone

$$X \succeq 0 \equiv c^* X c \geq 0 \quad \forall c \in \mathbb{C}^N$$



Valid Inequalities & Cuts (Mesh Gaps)

Outer Approximation of PSD cone

$$X \succeq 0 \equiv c^* X c \geq 0 \quad \forall c \in \mathbb{C}^N$$

Any *c* yields a necessary and hence *valid* inequality.

We would like to make a cut using *c* that **separates** an arbitrary non-PSD solution i.e. *renders it infeasible*.

Valid Inequalities & Cuts (Mesh Gaps)

Linear PSD cuts (linpsd)

Consider a relaxed non-PSD solution:

$$\hat{X} := \sum_i \lambda_i d_i d_i^*, \lambda_i \leq \lambda_{i+1}$$

Courant-Fischer-Weyl min-max principle:

$$\lambda_i = \max_{\dim(C)=i} \min_{c \in C: ||c||=1} c^* \hat{X} c$$

Max violation cut uses most negative eigenvalue:

(linpsd) $\lambda_n = \min_{c \in \mathbb{C}^n: ||c||=1} c^* \hat{X} c = d_1^* \hat{X} d_1$

(Sherali & Fraticelli, 2002)

SOC PSD cuts (socpsd)

$$X \succeq 0 \implies C^* X C \succeq 0$$

Setting C = [c1 c2] (n-by-2) results in SOC:

$$C^* X C \succeq 0 \equiv$$

$$\begin{bmatrix} c_1^* X c_1 & c_1^* X c_2 \\ c_2^* X c_1 & c_2^* X c_2 \end{bmatrix} \succeq 0 \equiv$$

$$c_1^* X c_1 \geq 0 \ (1)$$

$$c_2^* X c_2 \geq 0 \ (2)$$

$$(c_1^* X c_1)(c_2^* X c_2) \geq |c_1^* X c_2|^2 \ (3)$$

$$\equiv c_1^* X c_1 + c_2^* X c_2 \geq ||(2\mathrm{Re}(c_1^* X c_2), 2\mathrm{Im}(c_1^* X c_2), c_1^* X c_1 - c_2^* X c_2)||$$

(Oskoorouchi, Mitchell, 2006)

Valid Inequalities & Cuts (Mesh Gaps)

socpsd

For the SOC cuts we observe:
2x2 approach is a special case (e.g. RECSOCP)
linear cuts are dominated by these equalities

We propose:
socpsd: $C = [d_{n-1} \; d_n]$

Fishing for mesh gaps

Negative costs: $\min -\sum_i c_2 P_i$

Buses	Mesh gap
14	**2.1%**
30	**2.3%**
39	0.3%
57	0.7%
118	**1.1%**
300	0.6%

Valid Inequalities & Cuts (Mesh Gaps)

Cutting Plane Algorithm

1. Solve relaxation and obtain near-optimal solution
2. Generate cuts to separate violated constraints
3. Add cuts (linpsd/socpsd1/socpsd2) to constraint set
4. Purge cuts if optimal objective has improved
5. GOTO 1

Valid Inequalities & Cuts (Mesh Gaps)

Closing larger mesh gaps to 0.05%

	Iterations		Total time in solver (overhead not included)		Sparse RECSDP time to solve	Time to solve last iteration	
Buses	linsdp	socsdp	linsdp	socsdp		linsdp	socsdp
14	8	4	0.2	0.1	0.6	0.04	0.04
30	22	7	10.1	0.6	0.7	0.49	0.4
118	18	7	12.8	4.4	1.4	1.28	1
300	9	6	50.2	36.3	-	8	10

Summary of PSD cuts

PSD cuts are not ideal for precise solutions

- *Might* be competitive with standard interior point with a lot of engineering (Mitchell, 2003)

One clear use is when duality gap is significant

- Quickly establish lower bounds when branching
  - Improve if the pruning procedure demands it
- Fast separation procedures
  - Spatial disjunctions
  - Integers

Valid Inequalities & Cuts (Duality Gap)

Plumbing the depth of duality gap

Recall $\min -\sum_i c_2 P_i$

Buses	Duality gap	Mesh gap
9	<19%	0.02%
14	<18%	2.1%
30	<36%	2.3%
39	?	0.3%
57	?	0.7%
118	<23%	1.1%
300	?	0.6%

Valid Inequalities & Cuts (Duality Gap)

Global Optimization

Piecemeal approach to duality gap

- Add 'easy' nonconvex valid inequalities
  - 'Inner approximation'
  - e.g. univariate spatial disjunctions

(Phan, 2010): Zero duality gap, no branching

(Gopalakrishnan et al, 2012): Prove small duality gaps on modified IEEE cases (up to 2%)

Valid Inequalities & Cuts (Duality Gap)

Valid Spatial Disjunctions

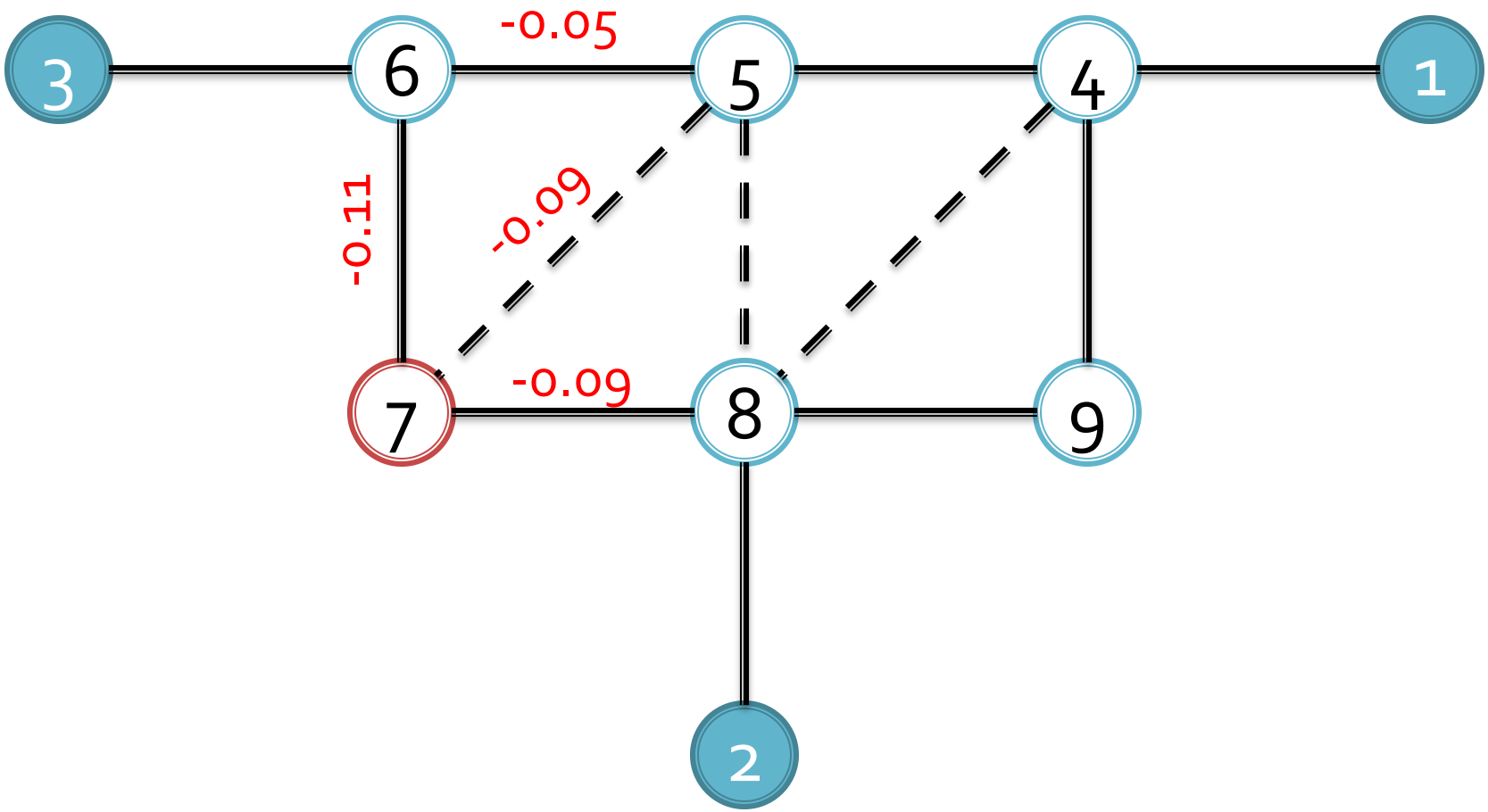
Selecting a disjunction:

$$\begin{bmatrix} W & T \\ -T & W \end{bmatrix} \succeq 0, \quad \boxed{||(2W_{ij}, 2T_{ij}, W_{ii} - W_{jj})|| \geq W_{ii} + W_{jj}}$$

e.g. find a violated line, add disjunction on voltage magnitude/angle

Valid Inequalities & Cuts (Duality Gap)

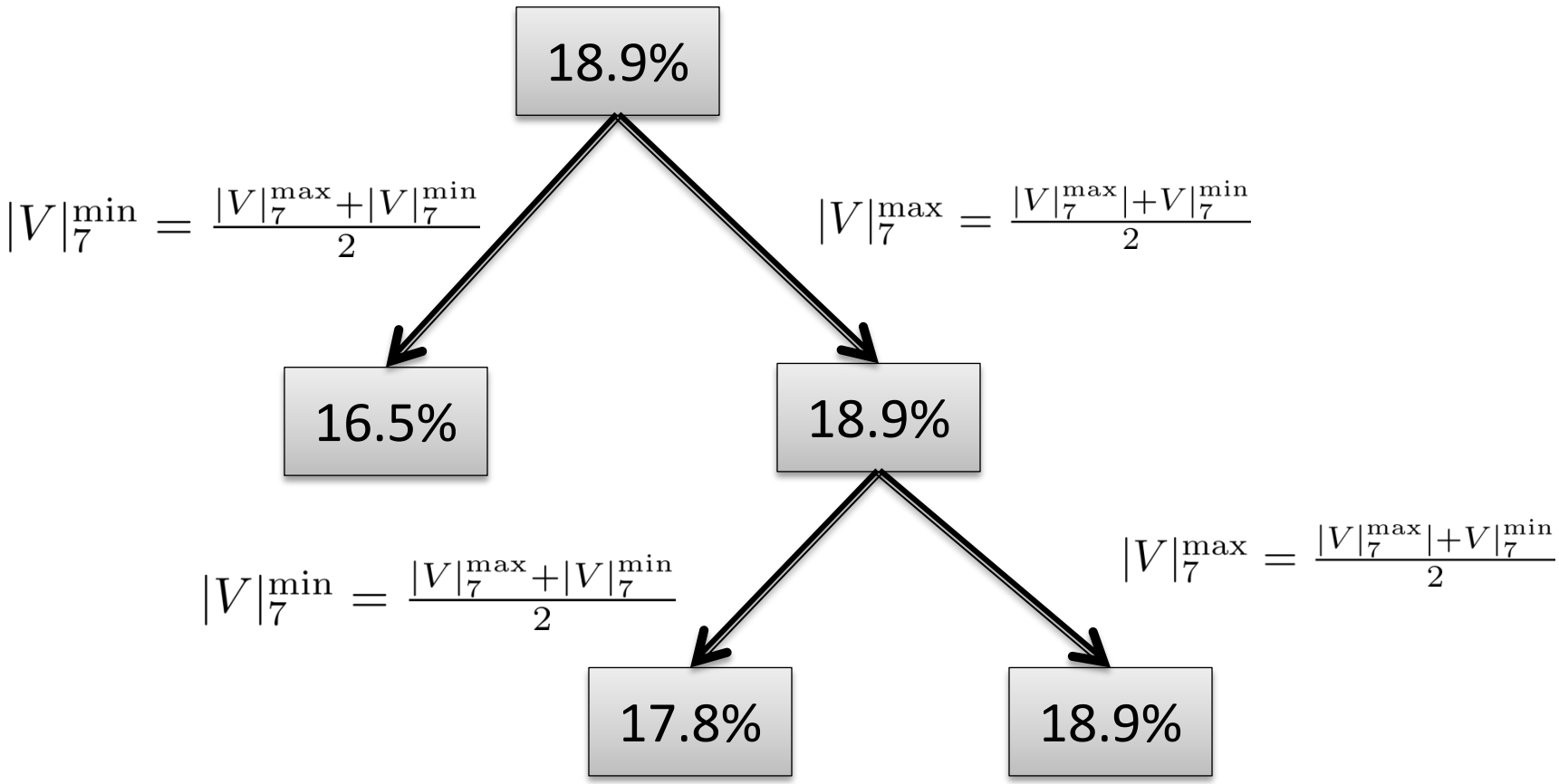
Example: negcase9



Valid Inequalities & Cuts (Duality Gap)

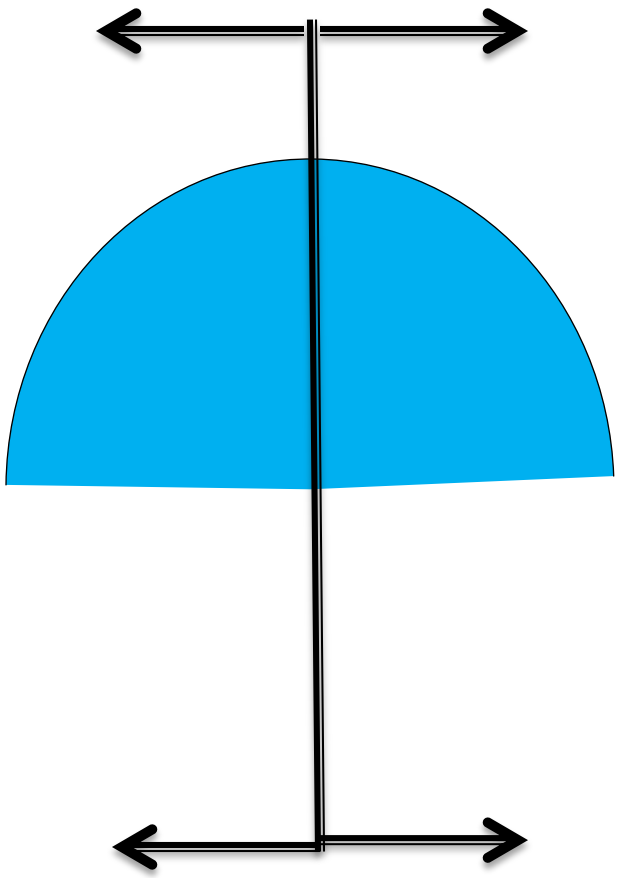
case9: Branching on flows

$$|V| \leq \frac{|V|_7^{\max}+|V|_7^{\min}}{2} \vee |V| \geq \frac{|V|_7^{\max}+|V|_7^{\min}}{2}$$



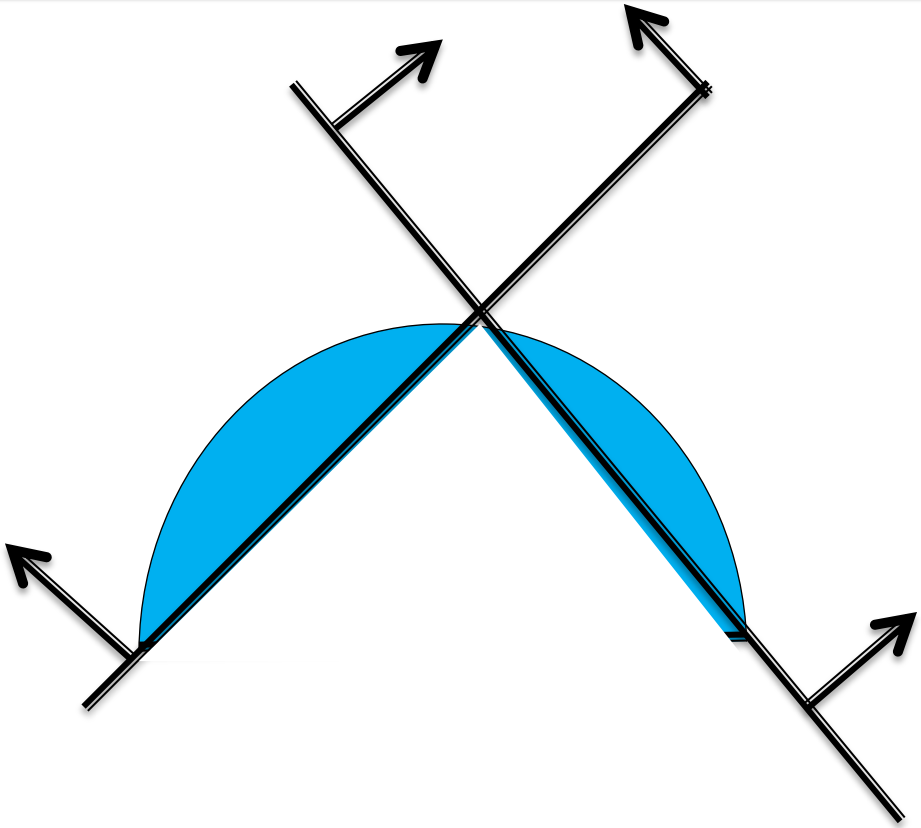
Valid Inequalities & Cuts (Duality Gap)

Feasible regions



Valid Inequalities & Cuts (Duality Gap)

Updating regions with valid ineqs



Valid Inequalities & Cuts (Duality Gap)

Convex overestimation

Best overestimate of convex function is a hyperplane

$$||(2W_{in}, 2T_{in}, W_{ii} - W_{nn})|| \leq \pi_0 + \pi_1 W_{in} + \pi_2 T_{in} + \pi_3 W_{ii} + \pi_4 W_{nn}$$

Given a convex space bounding the variables, we can use extreme point analysis to derive coefficients (secant approx)

Valid Inequalities & Cuts (Duality Gap)

Establishing variable bounds

$$\theta_{in}^{\min} \leq \theta_i - \theta_n \leq \theta_{in}^{\max}$$
$$V_i^{\min} \leq |V|_i \leq V_i^{\max}$$
$$\Longrightarrow$$
$$W_{in}^- = V_i^{\min} V_n^{\min} \min(\cos(\theta_{in}^{\min}), \cos(\theta_{in}^{\max}))$$
$$W_{in}^+ = \begin{cases} V_i^{\max} V_n^{\max}, & \theta^{\min}\theta^{\max} \leq 0 \\ V_i^{\max} V_n^{\max} \cos(\theta^{\min}), & \text{otherwise} \end{cases}$$
$$T_{in}^- = \begin{cases} V_i^{\max} V_n^{\max} \sin(\theta^{\min}), & \theta^{\min} \leq 0 \\ V_i^{\min} V_n^{\min} \sin(\theta^{\min}), & \text{otherwise} \end{cases}$$
$$T_{in}^+ = \begin{cases} V_i^{\max} V_n^{\max} \sin(\theta^{\max}), & \theta^{\max} \geq 0 \\ V_i^{\min} V_n^{\min} \sin(\theta^{\max}), & \text{otherwise} \end{cases}$$
$$W_{in}^- \leq W_{in} \leq W_{in}^+$$
$$T_{in}^- \leq T_{in} \leq T_{in}^+$$
$$(V_i^{\min})^2 \leq u_i \leq (V_i^{\max})^2$$

Valid Inequalities & Cuts (Duality Gap)
Overestimator

We update our secant valid inequality each time we branch on voltages

(**secvi**) $\pi_0 + \pi_1 W_{ij} + \pi_2 T_{ij} + \pi_3 W_{ii} + \pi_4 W_{jj} \geq W_{ii} + W_{jj}$

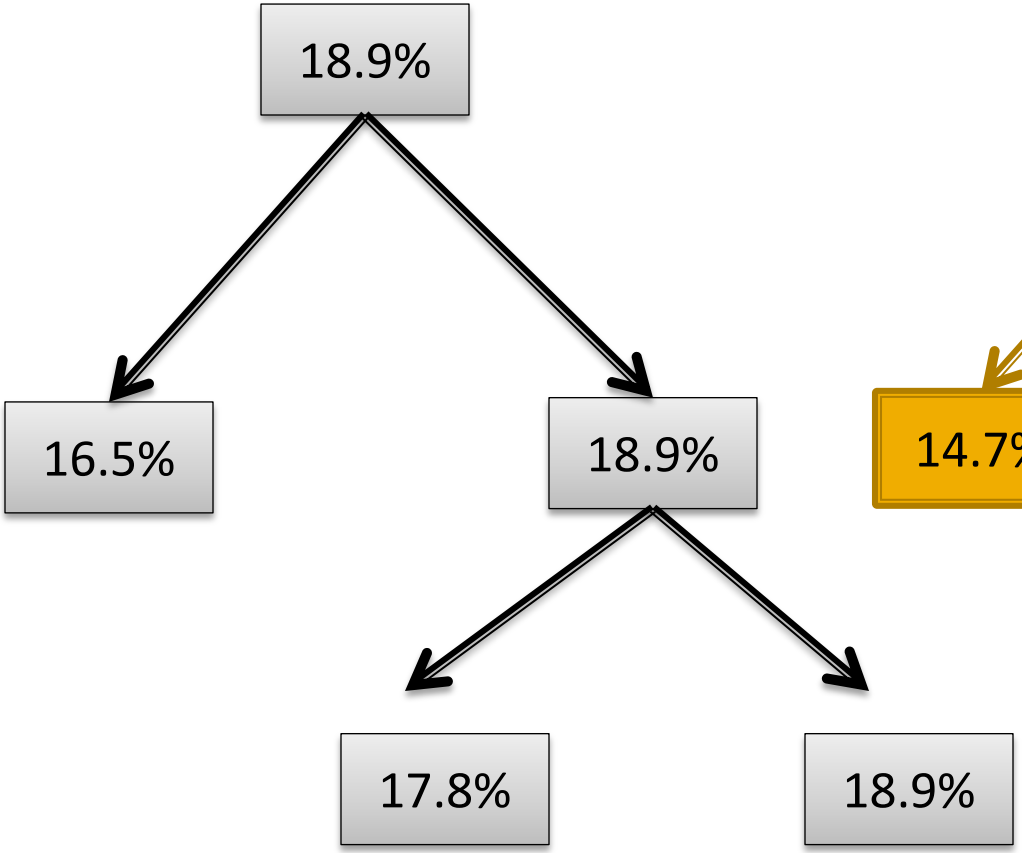
Enables branching on magnitudes or angles

Simple univariate secant if branching only on magnitudes due to norm symmetry

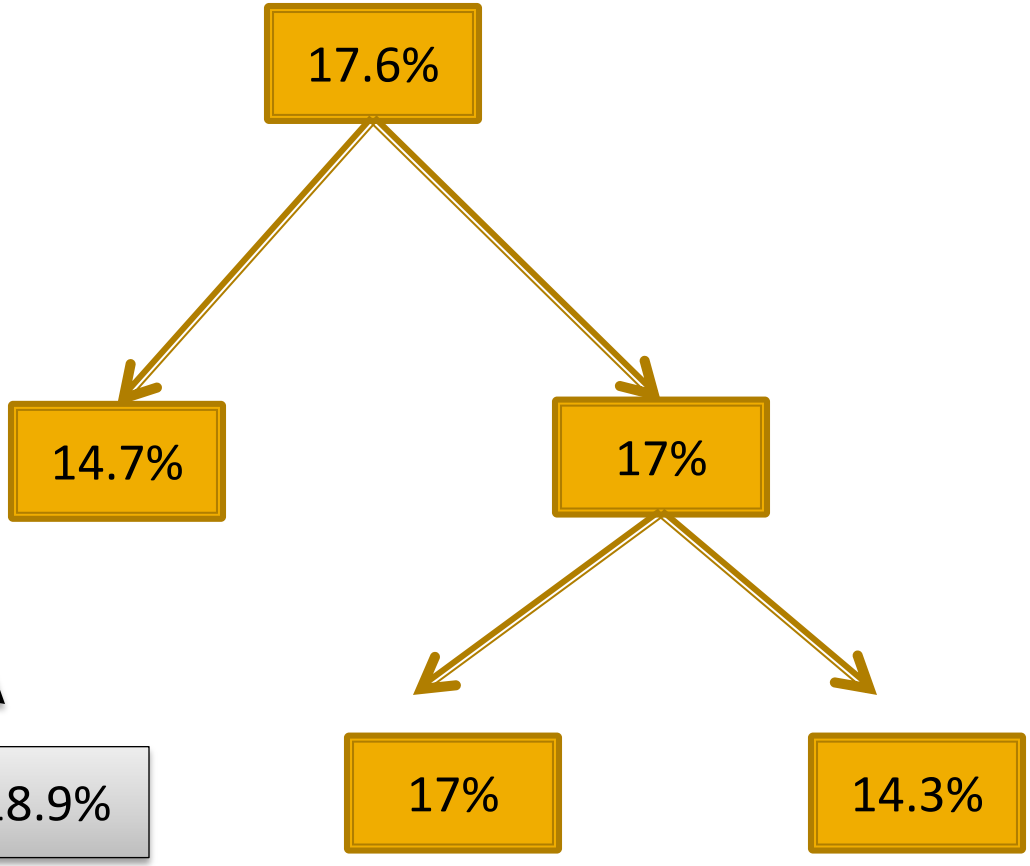
Valid Inequalities & Cuts (Duality Gap)

secvi on case9 (15 degs)

RECSDP



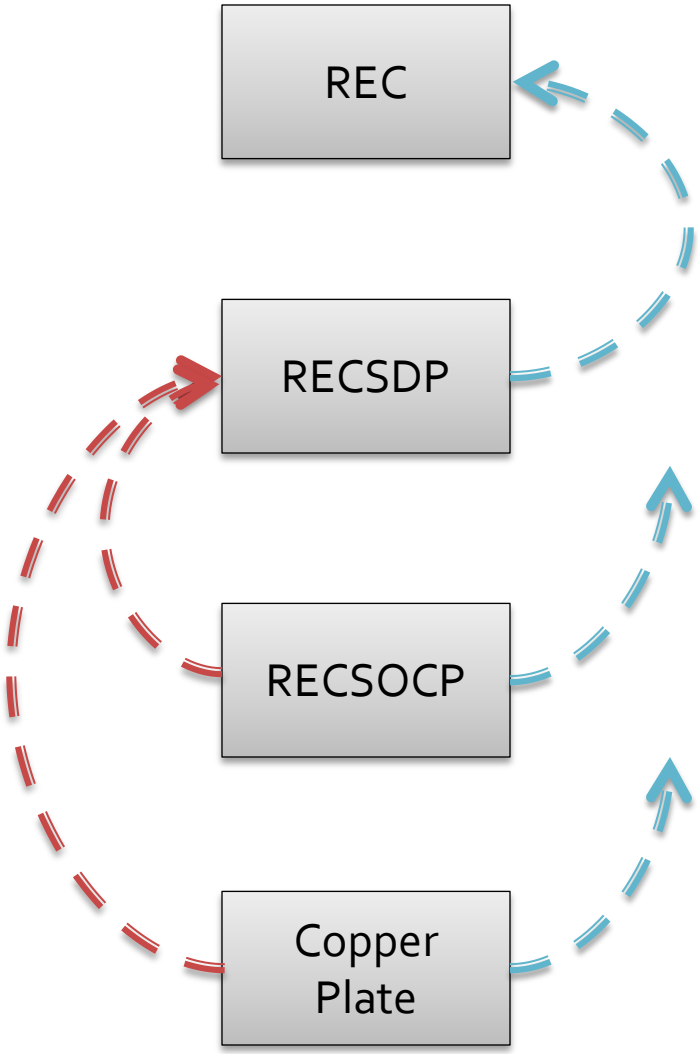
RECSDP+secvi



Valid Inequalities & Cuts

Cutting Planes and Disjunctions

Sparse PSD cuts get weaker relaxations to approach the Lagrangian Dual lower bound



Branching or disjunctive cuts get any relaxation to approach the true optimal value

Future Research

Selection of disjunctions/branching

- Can a feasible solution help guide lower bound construction?

SDP relaxation + cuts for cliques > 3?

Are there large mesh gaps? Duality gaps?

Relaxations in the original space of variables

- Projected RLT, SDP cuts on the IV formulation?

Conclusion

Cuts can help close root (mesh, duality) gaps

We can make arbitrarily effective outer-approximations of SDP in LP or SOCP form

Secant valid inequalities help global convergence in the case of duality gap

Thank you!

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Valid Inequalities & Cuts (Mesh Gaps)

Closing mesh gap to 0.05%

	Iterations		Total time in solvers (without overhead)		Sparse RECSDP time to solve	Time to solve last iteration	
Buses	linsdp	socsdp	linsdp	socsdp		linsdp	socsdp
57	6	3	1.8	1.4	0.5	0.4	0.6
118	12	5	9.8	6.2	1	1.3	1.8
300	9	6	50.2	36.3	-	8	10

Lagrangian Dual

$$\min x^T P_0 x + q_0^T x + r_0$$
$$\text{s.t.}$$
$$x^T P_i x + q_i^T x + r_i \leq 0, i = 1, ..., m$$

$$L(x, \lambda) = x^T (P_0 + \sum_{i=1}^{m} \lambda_i P_i) x + (q_0 + \sum_{i=1}^{m} \lambda_i q_i)^T x + r_0 + \sum_{i=1}^{m} \lambda_i r_i$$

$P \succeq 0, q \in \mathcal{R}(P) \implies \min_{x \in \mathbb{R}^n} x^T P x + q^T x + r = r - \frac{1}{4} q^T P^\dagger q$

Else $-\infty$

$$\implies \inf_{x \in \mathbb{R}^n} L(x, \lambda) = -\frac{1}{4}(q_0 + \sum_{i=1}^{m} \lambda_i q_i)^T (P_0 + \sum_{i=1}^{m} \lambda_i P_i)^\dagger (q_0 + \sum_{i=1}^{m} \lambda_i q_i) + \sum_{i=1}^{m} \lambda_i r_i + r_0$$

Adding substitution variable γ and using Schur complements, we can obtain the dual as an SDP.

Lagrangian Dual as SDP

$$\text{maximize } \gamma + \sum_{i=1}^{m} \lambda_i r_i + r_0$$

subject to

$$\begin{pmatrix} (P_0 + \sum_{i=1}^{m} \lambda_i P_i) & (q_0 + \sum_{i=1}^{m} \lambda_i q_i)/2 \\ (q_0 + \sum_{i=1}^{m} \lambda_i q_i)^T/2 & -\gamma \end{pmatrix} \succeq 0$$

$$\lambda \geq 0$$

Also the dual of Shor's relaxation!

Phase angle representation

Changing from polar to rectangular coordinates, we use:

$V = V^{Re} + jV^{Im}$, $V^{Re} = |V|\cos(\theta)$, and $V^{Re} = |V|\sin(\theta)$

The polar phase angle constraints (where it's understood the $\pm$ represents two inequalities):

$\pm(\theta_i - \theta_n) \leq \theta_{in}^{\max}$

Becomes these (which requires an assumption of nonzero angles at each bus):

$\pm(\arctan(\frac{V_i^{\mathrm{Im}}}{V_i^{\mathrm{Re}}}) - \arctan(\frac{V_n^{\mathrm{Im}}}{V_n^{\mathrm{Re}}})) \leq \theta_{in}^{\max}$

There is another way to obtain a rectangular constraint. For $\theta_{in}^{\max} < \frac{\pi}{2}$ we have these equivalent polar phase angle constraints:

$\pm\tan(\theta_i - \theta_n) \leq \tan(\theta_{in}^{\max})$

We can safely assume voltage magnitudes are nonzero so we can multiply both sides by them:

$\pm(|V_i||V_n|\sin(\theta_i - \theta_n)) \leq |V_i||V_n|\cos(\theta_i - \theta_n)\tan(\theta_{in}^{\max})$

This leads to the following rectangular constraints:

$\pm(V_i^{\mathrm{Re}}V_n^{\mathrm{Im}} - V_i^{\mathrm{Im}}V_n^{\mathrm{Re}}) \leq (V_i^{\mathrm{Re}}V_n^{\mathrm{Re}} + V_i^{\mathrm{Im}}V_n^{\mathrm{Im}})\tan(\theta_{in}^{\max})$

Note that one can also derive one rectangular constraint from the other using the identity $\arctan(x) - \arctan(y) = \arctan(\frac{x-y}{(1+xy)})$.

Valid Inequalities & Cuts (Mesh Gaps)

Reaching PSD feasibility

	Iterations		Total time in solvers (without overhead)		Sparse RECSDP time to solve	Time to solve last iteration	
Buses	linsdp	socsdp	linsdp	socsdp		linsdp	socsdp
57	22	8	12.7	5.5	0.5	1.9	0.9
118	27	9	32.8	12.9	1	1.1	1.8
300	17	8	118.8	61.2	-	12.4	21