Low-Rank Solution for Nonlinear optimization over AC Transmission Networks

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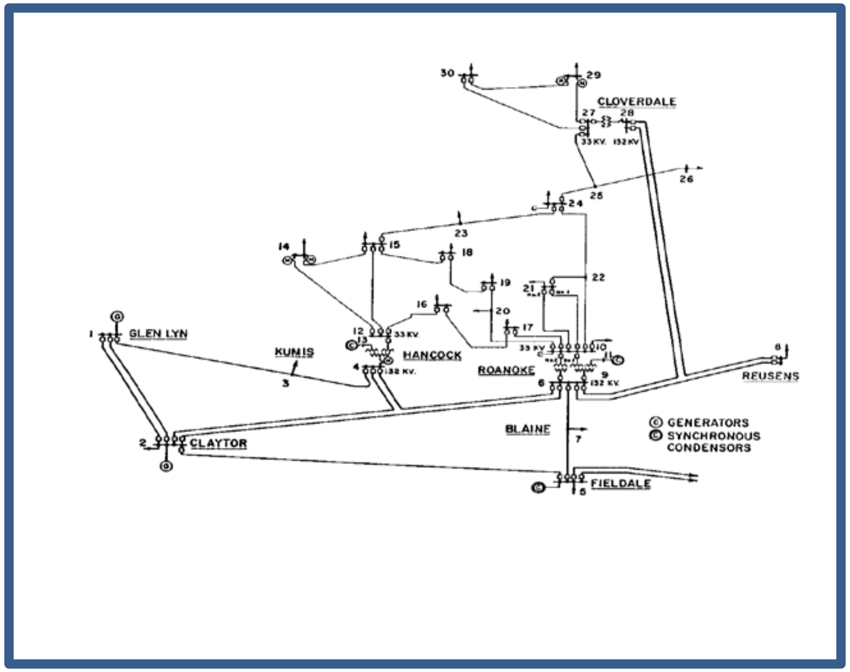
Power Networks

- **Optimizations:**
  - Optimal power flow (OPF)
  - Security-constrained OPF
  - State estimation
  - Network reconfiguration
  - Unit commitment
  - Dynamic energy management



- **Issue of non-convexity:**
  - Discrete parameters
  - Nonlinearity in continuous variables

- **Transition from traditional grid to smart grid:**
  - More variables (10X)
  - Time constraints (100X)

Nonlinear Optimizations

- OPF-based problems solved on different time scales:
  - Electricity market
  - Real-time operation
  - Security assessment
  - Transmission planning

- **Existing methods:** (i) linearization, (ii) local search

- **Question:** How to find the best solution using a scalable robust algorithm?

- **Approach:** Push all nonlinearities into a single rank constraint

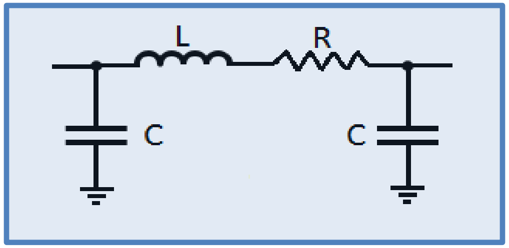
- **Applications:** (i) static/dynamic optimization, (ii) decentralized control

Old Results

(joint work with Steven Low, Somayeh Sojoudi, David Tse, Baosen Zhang, Stephen Boyd, Eric Chu and Matt Kranning)

Project 1: How to solve a given OPF in polynomial time?

- ❑ A sufficient condition to globally solve OPF:
  - Numerous randomly generated systems
  - IEEE systems with 14, 30, 57, 118, 300 buses
  - European grid



Project 2: Find network topologies over which optimization is easy?

- ❑ Distribution networks are fine.
- ❑ Every transmission network can be turned into a good one.

Project 3: How to design a distributed algorithm for solving OPF?

- ❑ A practical (infinitely) parallelizable algorithm
- ❑ It solves 10,000-bus OPF in 0.85 seconds on a single core machine.

New Results

(joint work with Somayeh Sojoudi and Ramtin Madani)

1- Optimization: How to do optimization over mesh networks?

2- Decentralized control: How to design an optimal distributed controller?

- **Approach:**

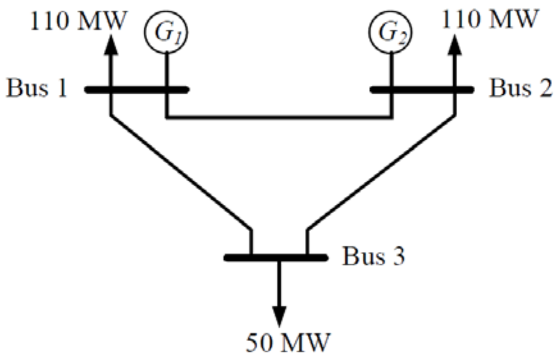
Quadratic optimization in x → Linear optimization in xx^T

↓

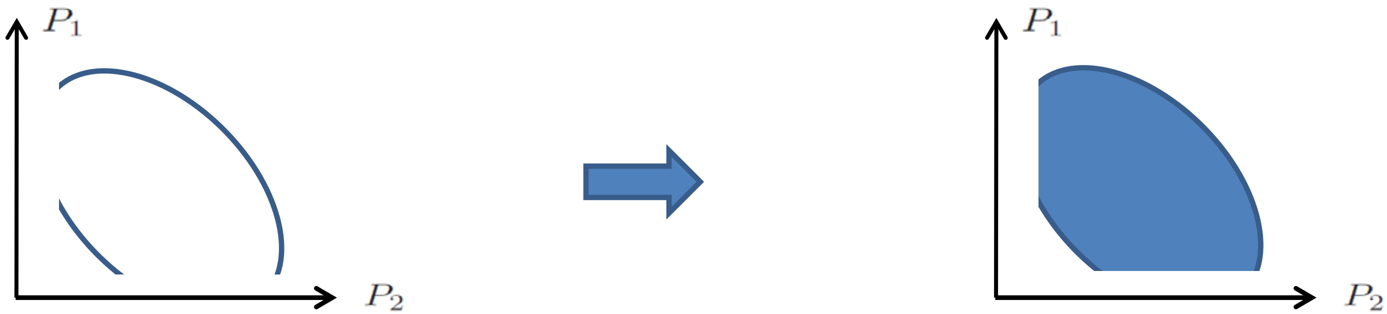
Replace xx^T with a matrix W subject to $W \succeq 0$ and $\text{rank}\{W\} = 1$

← Remove the rank constraint and penalize its effect

Geometric Intuition: Two-Generator Network



$$\text{minimize } f_1(P_1) + f_2(P_2)$$
$$\text{subject to } (P_1, P_2) \in \mathcal{P}$$



$$\text{minimize } f_1(P_1) + f_2(P_2)$$
$$\text{subject to } (P_1, P_2) \in \mathcal{P}$$

$$=$$

$$\text{minimize } f_1(P_1) + f_2(P_2)$$
$$\text{subject to } (P_1, P_2) \in \text{conv}(\mathcal{P})$$

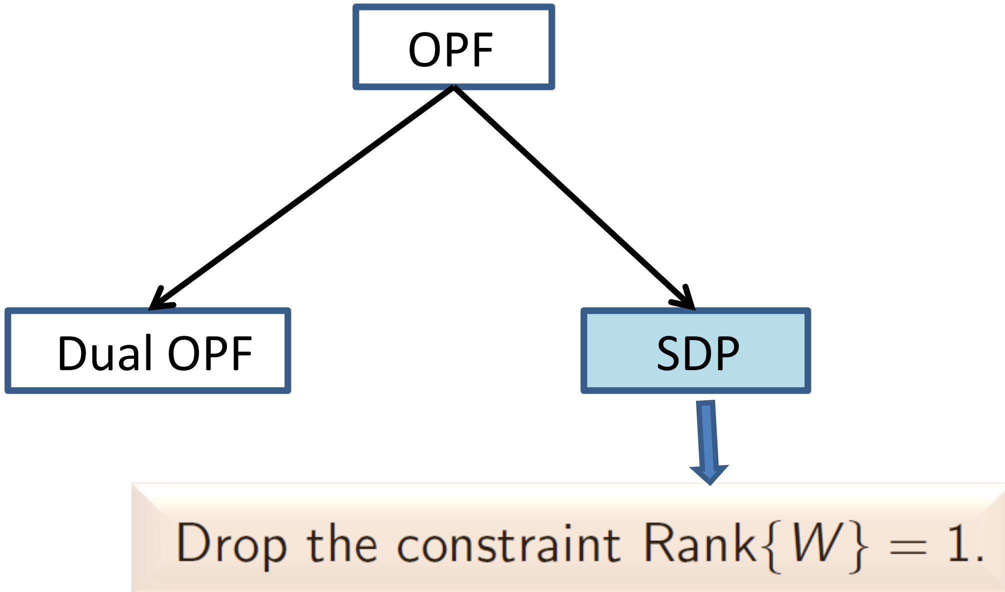
Optimal Power Flow

$$
\begin{aligned}
\min_{\mathbf{V},\mathbf{P}_G,\mathbf{Q}_G} \quad & \sum_{k\in\mathcal{G}} f_k(P_{G_k}) && \text{(1a)}\\
\text{Subject to} \quad & P_k^{\min} \leq P_{G_k} \leq P_k^{\max} && \text{(1b)}\\
& Q_k^{\min} \leq Q_{G_k} \leq Q_k^{\max} && \text{(1c)}\\
& V_k^{\min} \leq |V_k| \leq V_k^{\max} && \text{(1d)}\\
& \mathrm{Re}\left\{V_l(V_l - V_m)^* y_{lm}^*\right\} \leq P_{lm}^{\max} && \text{(1e)}\\
& \mathrm{trace}\{\mathbf{V}\mathbf{V}^*\mathbf{Y}^* e_k e_k^*\} = P_{G_k} - P_{D_k} + (Q_{G_k} - Q_{D_k})\mathrm{i} && \text{(1f)}
\end{aligned}
$$

- (1a) → **Cost**
- (1b)–(1d) → **Operation**
- (1e) → **Flow**
- (1f) → **Balance**

Trick: Replace $\mathbf{VV}^*$ with a matrix $\mathbf{W} \succeq 0$ subject to $\mathrm{rank}\{\mathbf{W}\} = 1$.

Various Relaxations



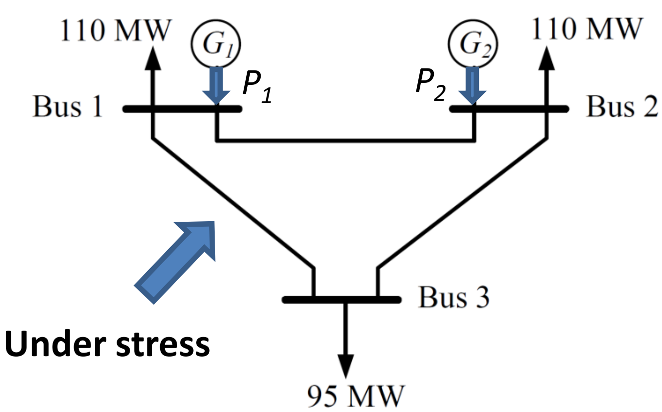
- SDP relaxation:
  - IEEE systems
  - SC Grid
  - European grid
  - Random systems

- Exactness of SDP relaxation and zero duality gap are equivalent for OPF.

Theorem

Exact relaxation for DC/AC distribution and DC transmission networks.

Response of SDP to Equivalent Formulations

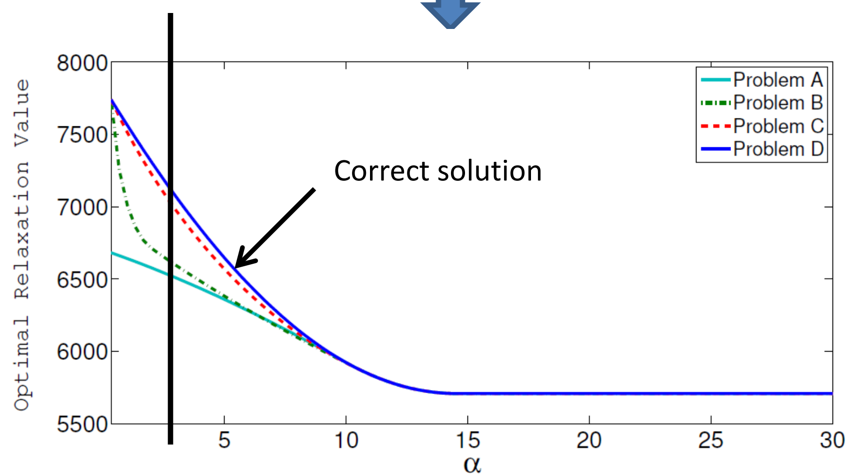


1. Equivalent formulations behave differently after relaxation.

2. SDP works for weakly-cyclic networks with cycles of size 3 if voltage difference is used to restrict flows.

❑ **Capacity constraint:** active power, apparent power, angle difference, voltage difference, current?

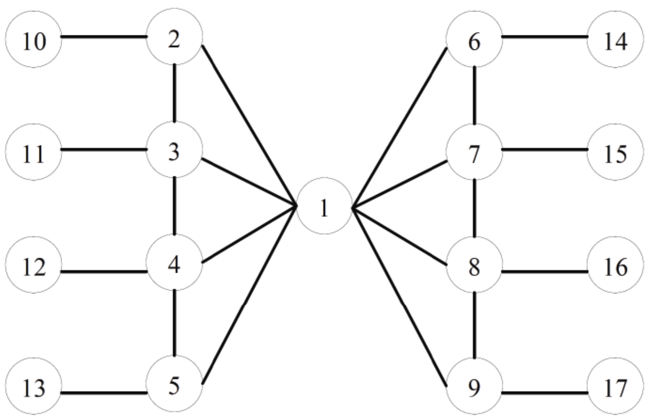
Problem A :	$\theta_{13}^{\max} = \alpha$
Problem B :	$P_{13}^{\max} = \mathrm{Re}\{(1 - e^{\alpha i}) y_{13}^*\}$
Problem C :	$S_{13}^{\max} = \lvert (1 - e^{\alpha i}) y_{13}^* \rvert$
Problem D :	$\Delta V_{13}^{\max} = 2\,(1 - \cos(\alpha))$



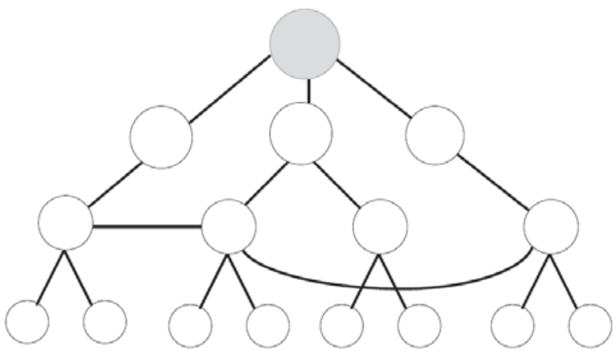
Low-Rank Solution

Definition

Define η as the minimum number of vertices whose removal from the power network eliminates all cycles of the network.



$$\eta = 1$$



Theorem

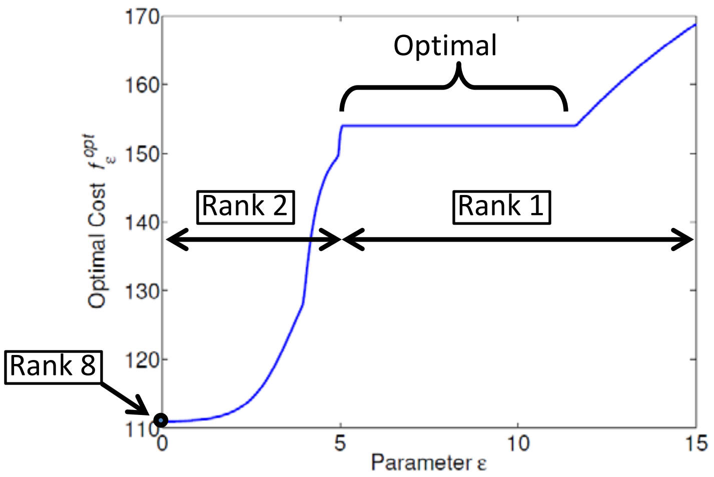
If OPF is feasible, then its relaxation has a solution $(\mathbf{W}^{\text{opt}}, \mathbf{P}_G^{\text{opt}}, \mathbf{Q}_G^{\text{opt}})$ *such that* $\text{rank}\{\mathbf{W}^{\text{opt}}\} \leq \eta + 1$.

Penalization of Rank Constraint

- **How to turn a low-rank solution into a rank-1 solution?**

- Perturbed SDP relaxation:

$$\sum_{k\in\mathcal{G}} f_k(P_{G_k}) \Rightarrow \sum_{k\in\mathcal{G}} f_k(P_{G_k}) - \varepsilon \sum_{(l,m)\in\mathcal{L}} \text{Re}\{W_{lm}\}$$

10-bus cycle (case I)	10-bus cycle (case II)
o Eigs of ***W*** for *ε=0:* $0.0132,\ 0.0146,\ 0.0381,\ 0.0694,\ 0.0896,\ 0.2134,\ 0.3167,\ 0.5424,\ 1.4405,\ 7.3939$ o Eigs of ***W*** for *ε=10*$^{-5}$*:* $0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 10.5$	

Decentralized Control Problem as Low-Rank Optimization

- Consider the dynamical system: $\begin{cases} x[\tau+1] = Ax[\tau] + Bu[\tau] \\ y[\tau] = Cx[\tau] \end{cases}, \quad \tau = 0, 1, 2, \ldots$

- Optimal decentralized control (ODC) problem:

Controller	Objective function	Box constraints
$u[\tau] = Ky[\tau]$ (diagonal K)	$\sum_{\tau=0}^{p} \left(x[\tau]^T Q[\tau] x[\tau] + u[\tau]^T R[\tau] u[\tau] \right)$	$\underline{\alpha}[\tau] \leq x[\tau] \leq \overline{\alpha}[\tau]$ $\underline{\beta}[\tau] \leq u[\tau] \leq \overline{\beta}[\tau]$

- ODC is a quadratic optimization.

Decentralized Control Problem as Low-Rank Optimization

- Define a vector of variables:

$$v = \begin{bmatrix} 1 & \underbrace{\text{vec}\{K\}}_{\text{nonzero entries of } K} & \underbrace{x[0]^T \cdots x[p]^T}_{\text{state}} & \underbrace{y[0]^T \cdots y[p]^T}_{\text{output}} & \underbrace{u[0]^T \cdots u[p]^T}_{\text{input}} \end{bmatrix}^T$$

- ODC is a linear optimization in terms of vv^T.

Definition

Convexified ODC: Replace vv^T with a positive semidefinite matrix $\mathbf{W}$ in the reformulated ODC.

Theorem

Convexified ODC has a solution $\mathbf{W}^{opt}$ *with rank at most 4.*

Integrated OPF + Dynamics

- Synchronous machine with interval voltage $|E|e^{j\delta}$ and terminal voltage $|V|e^{j\theta}$.

- **Swing equation:**

$$\frac{d\delta(t)}{dt} = \omega(t)$$

$$\frac{d(|E|\sin(\delta(t)))}{dt} = |E|\cos(\delta(t))\omega(t)$$

$$M\frac{d\omega}{dt} = -D\omega(t) + P_M(t) - \frac{|E||V(t)|\sin(\delta(t)-\theta(t))}{\alpha}$$

- **Define:** $\mathbf{x}(t) = \left[\ 1 \quad \omega(t) \quad \mathrm{Re}\{E\} \quad \mathrm{Im}\{E\} \quad \mathrm{Re}\{V(t)\} \quad \mathrm{Im}\{V(t)\}\ \right]^H$

- **Linear system:**

$$\frac{dW_{14}(t)}{dt} = W_{32}(t)$$

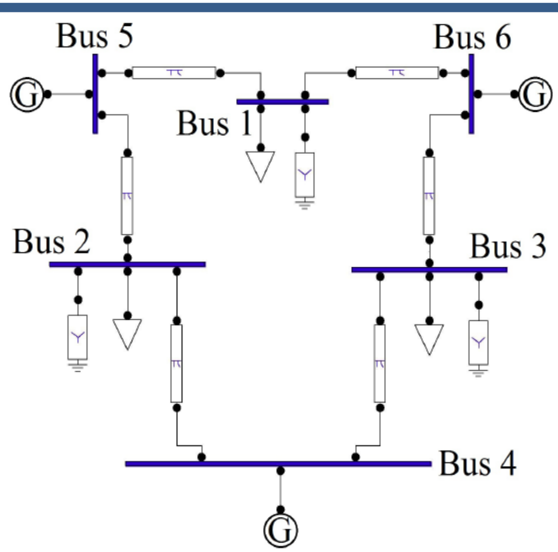
$$\frac{dW_{12}(t)}{dt} = -\frac{D}{M}W_{12}(t) - \frac{1}{M\alpha}(W_{45}(t) - W_{36}(t)) + \frac{1}{M}P_M(t)$$

Conclusions

❑ **Focus:**

❖ Nonlinear optimization

❖ Decentralized control



❑ Developed a low-rank optimization method

❑ Developed various theories to support the method