Low-Rank Solution for Nonlinear optimization over AC Transmission Networks

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Joint work with Somayeh Sojoudi and Ramtin Madani
Optimizations:
- Optimal power flow (OPF)
- Security-constrained OPF
- State estimation
- Network reconfiguration
- Unit commitment
- Dynamic energy management

Issue of non-convexity:
- Discrete parameters
- Nonlinearity in continuous variables

Transition from traditional grid to smart grid:
- More variables (10X)
- Time constraints (100X)
Nonlinear Optimizations

- OPF-based problems solved on different time scales:
  - Electricity market
  - Real-time operation
  - Security assessment
  - Transmission planning

- **Existing methods:** (i) linearization, (ii) local search

- **Question:** How to find the best solution using a scalable robust algorithm?

- **Approach:** Push all nonlinearities into a single rank constraint

- **Applications:** (i) static/dynamic optimization, (ii) decentralized control
Old Results
(joint work with Steven Low, Somayeh Sojoudi, David Tse, Baosen Zhang, Stephen Boyd, Eric Chu and Matt Kranning)

Project 1: How to solve a given OPF in polynomial time?

- A sufficient condition to globally solve OPF:
  - Numerous randomly generated systems
  - IEEE systems with 14, 30, 57, 118, 300 buses
  - European grid

Project 2: Find network topologies over which optimization is easy?

- Distribution networks are fine.
- Every transmission network can be turned into a good one.

Project 3: How to design a distributed algorithm for solving OPF?

- A practical (infinitely) parallelizable algorithm
- It solves 10,000-bus OPF in 0.85 seconds on a single core machine.
New Results
(joint work with Somayeh Sojoudi and Ramtin Madani)

1- Optimization: How to do optimization over mesh networks?

2- Decentralized control: How to design an optimal distributed controller?

Approach:

Quadratic optimization in $x$  \rightarrow Linear optimization in $xx^T$

Remove the rank constraint and penalize its effect  \rightarrow Replace $xx^T$ with a matrix $W$ subject to $W \succeq 0$ and $\text{rank}\{W\} = 1$
Geometric Intuition: Two-Generator Network

minimize $f_1(P_1) + f_2(P_2)$
subject to $(P_1, P_2) \in \mathcal{P}$

minimize $f_1(P_1) + f_2(P_2)$
subject to $(P_1, P_2) \in \text{conv} \mathcal{P}$
Optimal Power Flow

\[
\begin{align*}
\min_{\mathbf{V}, P_G, Q_G} & \quad \sum_{k \in G} f_k(P_{G_k}) \\
\text{Subject to} & \quad P_{k}^\text{min} \leq P_{G_k} \leq P_{k}^\text{max} \\
& \quad Q_{k}^\text{min} \leq Q_{G_k} \leq Q_{k}^\text{max} \\
& \quad V_{k}^\text{min} \leq |V_k| \leq V_{k}^\text{max} \\
& \quad \text{Re} \{V_l(V_l - V_m)^* y_{lm}^*\} \leq P_{lm}^\text{max} \\
& \quad \text{trace}(\mathbf{VV}^* \mathbf{Y}^* e_k e_k^*) = P_{G_k} - P_{D_k} + (Q_{G_k} - Q_{D_k})i
\end{align*}
\] (1a) (1b) (1c) (1d) (1e) (1f)

**Trick:** Replace \( \mathbf{VV}^* \) with a matrix \( \mathbf{W} \geq 0 \) subject to \( \text{rank}\{\mathbf{W}\} = 1 \).
Various Relaxations

- **SDP relaxation:**
  - IEEE systems
  - SC Grid
  - European grid
  - Random systems

- Exactness of SDP relaxation and zero duality gap are equivalent for OPF.

**Theorem**

*Exact relaxation for DC/AC distribution and DC transmission networks.*
Response of SDP to Equivalent Formulations

1. Equivalent formulations behave differently after relaxation.
2. SDP works for weakly-cyclic networks with cycles of size 3 if voltage difference is used to restrict flows.

- **Capacity constraint**: active power, apparent power, angle difference, voltage difference, current?

<table>
<thead>
<tr>
<th>Problem</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\theta_{13}^{\text{max}} = \alpha$</td>
</tr>
<tr>
<td>B</td>
<td>$P_{13}^{\text{max}} = \text{Re}{ (1 - e^{\alpha i}) y_{13}^* }$</td>
</tr>
<tr>
<td>C</td>
<td>$S_{13}^{\text{max}} =</td>
</tr>
<tr>
<td>D</td>
<td>$\Delta V_{13}^{\text{max}} = 2 \left( 1 - \cos(\alpha) \right)$</td>
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Correct solution
**Definition**

Define $\eta$ as the minimum number of vertices whose removal from the power network eliminates all cycles of the network.

**Theorem**

*If OPF is feasible, then its relaxation has a solution $(W^{\text{opt}}, P_G^{\text{opt}}, Q_G^{\text{opt}})$ such that $\text{rank}\{W^{\text{opt}}\} \leq \eta + 1$.***
Penalization of Rank Constraint

- How to turn a low-rank solution into a rank-1 solution?

Perturbed SDP relaxation:

\[ \sum_{k \in \mathcal{G}} f_k(P_{G_k}) \quad \rightarrow \quad \sum_{k \in \mathcal{G}} f_k(P_{G_k}) - \varepsilon \sum_{(l,m) \in \mathcal{L}} \text{Re}\{W_{lm}\} \]

### 10-bus cycle (case I)

- Eigs of \( W \) for \( \varepsilon = 0 \):
  
  0.0132, 0.0146, 0.0381, 0.0694, 0.0896, 
  0.2134, 0.3167, 0.5424, 1.4405, 7.3939

- Eigs of \( W \) for \( \varepsilon = 10^{-5} \):
  
  0, 0, 0, 0, 0, 0, 0, 0, 0, 10.5
Consider the dynamical system:

\[
\begin{align*}
    x[\tau + 1] &= Ax[\tau] + Bu[\tau] \\
    y[\tau] &= Cx[\tau]
\end{align*}
\]

, \quad \tau = 0, 1, 2, \ldots

Optimal decentralized control (ODC) problem:

<table>
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<tr>
<th>Controller</th>
<th>Objective function</th>
<th>Box constraints</th>
</tr>
</thead>
</table>
| \(u[\tau] = Ky[\tau]\) (diagonal \(K\)) | \[
\sum_{\tau=0}^{P} \left( x[\tau]^T Q[\tau] x[\tau] + u[\tau]^T R[\tau] u[\tau] \right) \] | \[
\alpha[\tau] \leq x[\tau] \leq \bar{\alpha}[\tau] \\
\beta[\tau] \leq u[\tau] \leq \bar{\beta}[\tau]
\] |

ODC is a quadratic optimization.
Decentralized Control Problem as Low-Rank Optimization

- Define a vector of variables:

\[ \nu = \begin{bmatrix} 1 & \text{vec}\{K\} & x[0]^T & \cdots & x[p]^T & y[0]^T & \cdots & y[p]^T & u[0]^T & \cdots & u[p]^T \end{bmatrix}^T \]

- ODC is a linear optimization in terms of \( \nu \nu^T \).

**Definition**

**Convexified ODC:** Replace \( \nu \nu^T \) with a positive semidefinite matrix \( \mathbf{W} \) in the reformulated ODC.

**Theorem**

*Convexified ODC has a solution* \( \mathbf{W}^{\text{opt}} \) *with rank at most 4.*
Integrated OPF + Dynamics

- Synchronous machine with interval voltage $|E|e^{j\delta}$ and terminal voltage $|V|e^{j\theta}$.

- Swing equation:
  \[
  \frac{d\delta(t)}{dt} = \omega(t)
  \]
  \[
  M \frac{d\omega(t)}{dt} = -D\omega(t) + P_M(t) - \frac{|E||V(t)|\sin(\delta(t)) - \theta(t))}{\alpha}
  \]

- Define:
  \[
  x(t) = \begin{bmatrix} 1 & \omega(t) & \text{Re}\{E\} & \text{Im}\{E\} & \text{Re}\{V(t)\} & \text{Im}\{V(t)\} \end{bmatrix}^H
  \]

- Linear system:
  \[
  \frac{dW_{14}(t)}{dt} = W_{32}(t)
  \]
  \[
  \frac{dW_{12}(t)}{dt} = -\frac{D}{M}W_{12}(t) - \frac{1}{M\alpha}(W_{45}(t) - W_{36}(t)) + \frac{1}{M}P_M(t)
  \]
Conclusions

- **Focus:**
  - Nonlinear optimization
  - Decentralized control

- Developed a low-rank optimization method
- Developed various theories to support the method