Application of Semidefinite Programming to Large-Scale Optimal Power Flow Problems

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Outline

- Introduction
- Application of semidefinite programming to the OPF problem
- Techniques for large OPF problems
- Results
- Conclusion

The Power Flow Equations

- Nonlinear, coupled quadratic form
- Solved using *locally* convergent techniques dependent on an initial guess of solution voltages

Polar voltage coordinates: $\vec{V}_i = V_i \, e^{j\delta_i}$

$$P_i = V_i \sum_{k=1}^{n} V_k \left(\mathrm{G}_{ik} \cos\left(\delta_i - \delta_k\right) + \mathrm{B}_{ik} \sin\left(\delta_i - \delta_k\right) \right)$$

$$Q_i = V_i \sum_{k=1}^{n} V_k \left(\mathrm{G}_{ik} \sin\left(\delta_i - \delta_k\right) - \mathrm{B}_{ik} \cos\left(\delta_i - \delta_k\right) \right)$$

Rectangular voltage coordinates: $\vec{V}_i = V_{di} + jV_{qi}$

$$P_i = V_{di} \sum_{k=1}^{n} \left(\mathrm{G}_{ik} V_{dk} - \mathrm{B}_{ik} V_{qk} \right) + V_{qi} \sum_{k=1}^{n} \left(\mathrm{B}_{ik} V_{dk} + \mathrm{G}_{ik} V_{qk} \right)$$

$$Q_i = V_{di} \sum_{k=1}^{n} \left(-\mathrm{B}_{ik} V_{dk} - \mathrm{G}_{ik} V_{qk} \right) + V_{qi} \sum_{k=1}^{n} \left(\mathrm{G}_{ik} V_{dk} - \mathrm{B}_{ik} V_{qk} \right)$$

$$V_i^2 = V_{di}^2 + V_{qi}^2$$

Optimal Power Flow (OPF) Problem

- Optimization used to determine system operation
  - Minimize generation cost while satisfying physical laws and engineering constraints
  - Yields generator dispatches, line flows, etc.
- Large scale
  - Optimize dispatch for several states
- Non-convex, NP-hard
- Studied for 40 years
  - Existing methods do not guarantee global optimum

Semidefinite Programming

- Convex optimization, finds *global optimum*

$$\min_{\mathbf{W}} \ \mathrm{trace}\left(\mathbf{BW}\right)$$

$$\text{subject to}$$

$$\mathrm{trace}\left(\mathbf{A}_i\mathbf{W}\right) \leq c_i$$

$$\mathbf{W} \succeq 0$$

Recall $\mathrm{trace}\left(\mathbf{A}^T\mathbf{W}\right) = A_{11}W_{11} + A_{12}W_{12} + \ldots + A_{nn}W_{nn}$

$$\mathbf{W} \succeq 0 \ \text{ if and only if } \ \mathrm{eig}\left(\mathbf{W}\right) \geq 0$$

Semidefinite Relaxation of the Power Flow Equations

- Write power flow equations as $x^T \mathbf{A}_i x = c_i$
  where $x = \begin{bmatrix} V_{d1} & V_{d2} & \cdots & V_{dn} & V_{q1} & V_{q2} & \cdots & V_{qn} \end{bmatrix}^T$
- Define matrix $\mathbf{W} = xx^T$
- Rewrite as $\mathrm{trace}\left(\mathbf{A}_i \mathbf{W}\right) = c_i$ and $\mathrm{rank}\left(\mathbf{W}\right) = 1$
- Relaxation:
  replace $\mathrm{rank}\left(\mathbf{W}\right) = 1$ with $\mathbf{W} \succeq 0$
  - $\mathrm{rank}\left(\mathbf{W}\right) = 1$ implies zero duality gap ("tight" relaxation) and recovery of the globally optimal voltage profile [Lavaei '12]

Example

$$x = \begin{bmatrix} V_{d1} & V_{d2} & \dots & V_{dn} & V_{q1} & V_{q2} & \dots & V_{qn} \end{bmatrix}^T$$

$$\mathrm{trace}\left(\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}}^{\mathbf{M}_1} \overbrace{\begin{bmatrix} V_{d1}^2 & V_{d1}V_{d2} & \cdots & V_{d1}V_{q1} & \cdots & V_{d1}V_{qn} \\ V_{d1}V_{d2} & V_{d2}^2 & \cdots & V_{d2}V_{q1} & \cdots & V_{d2}V_{qn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ V_{d1}V_{q1} & V_{d2}V_{q1} & \cdots & V_{q1}^2 & \cdots & V_{q1}V_{qn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ V_{d1}V_{qn} & V_{d2}V_{qn} & \cdots & V_{q1}V_{qn} & \cdots & V_{qn}^2 \end{bmatrix}}^{\mathbf{W} = xx^T} \right)$$

$$= V_{d1}^2 + V_{q1}^2 = V_1^2$$

Application of Semidefinite Programming to the Optimal Power Flow Problem

Classical OPF Problem

Cost

$$\min \sum_{k\in\mathcal{G}} f_k\left(P_{Gk}\right) = \sum_{k\in\mathcal{G}} c_{2k}P_{Gk}^2 + c_{1k}P_{Gk} + c_{0k}$$

Engineering Constraints

subject to

$$P_{Gk}^{\min} \le P_{Gk} \le P_{Gk}^{\max}$$

$$Q_{Gk}^{\min} \le Q_{Gk} \le Q_{Gk}^{\max}$$

$$\left(V_k^{\min}\right)^2 \le V_{dk}^2 + V_{qk}^2 \le \left(V_k^{\max}\right)^2$$

$$|S_{lm}| \le S_{lm}^{\max}$$

Physical Laws

$$P_{Gk} - P_{Dk} = V_{dk}\sum_{i=1}^{n}\left(\mathrm{G}_{ik}V_{di} - \mathrm{B}_{ik}V_{qi}\right) + V_{qk}\sum_{i=1}^{n}\left(\mathrm{B}_{ik}V_{di} + \mathrm{G}_{ik}V_{qi}\right)$$

$$Q_{Gk} - Q_{Dk} = V_{dk}\sum_{i=1}^{n}\left(-\mathrm{B}_{ik}V_{di} - \mathrm{G}_{ik}V_{qi}\right) + V_{qk}\sum_{i=1}^{n}\left(\mathrm{G}_{ik}V_{di} - \mathrm{B}_{ik}V_{qi}\right)$$

Semidefinite Relaxation of the OPF Problem

$$\min_{\mathbf{W}} \quad \sum_{k \in \mathcal{G}} c_{k2} \left(\mathrm{trace}\left(\mathbf{Y}_k \mathbf{W}\right) + P_{Dk}\right)^2 + c_{k1} \left(\mathrm{trace}\left(\mathbf{Y}_k \mathbf{W}\right) + P_{Dk}\right) + c_{k0}$$

$$\text{subject to} \qquad P_{Gk}^{min} - P_{Dk} \leq \mathrm{trace}\left(\mathbf{Y}_k \mathbf{W}\right) \leq P_{Gk}^{max} - P_{Dk}$$

$$Q_{Gk}^{min} - Q_{Dk} \leq \mathrm{trace}\left(\bar{\mathbf{Y}}_k \mathbf{W}\right) \leq Q_{Gk}^{max} - Q_{Dk}$$

$$\left(V_k^{min}\right)^2 \leq \mathrm{trace}\left(\mathbf{M}_k \mathbf{W}\right) \leq \left(V_k^{max}\right)^2$$

$$\mathrm{trace}\left(\mathbf{Y}_{lm} \mathbf{W}\right)^2 + \mathrm{trace}\left(\bar{\mathbf{Y}}_{lm} \mathbf{W}\right)^2 \leq \left(S_{lm}^{max}\right)^2$$

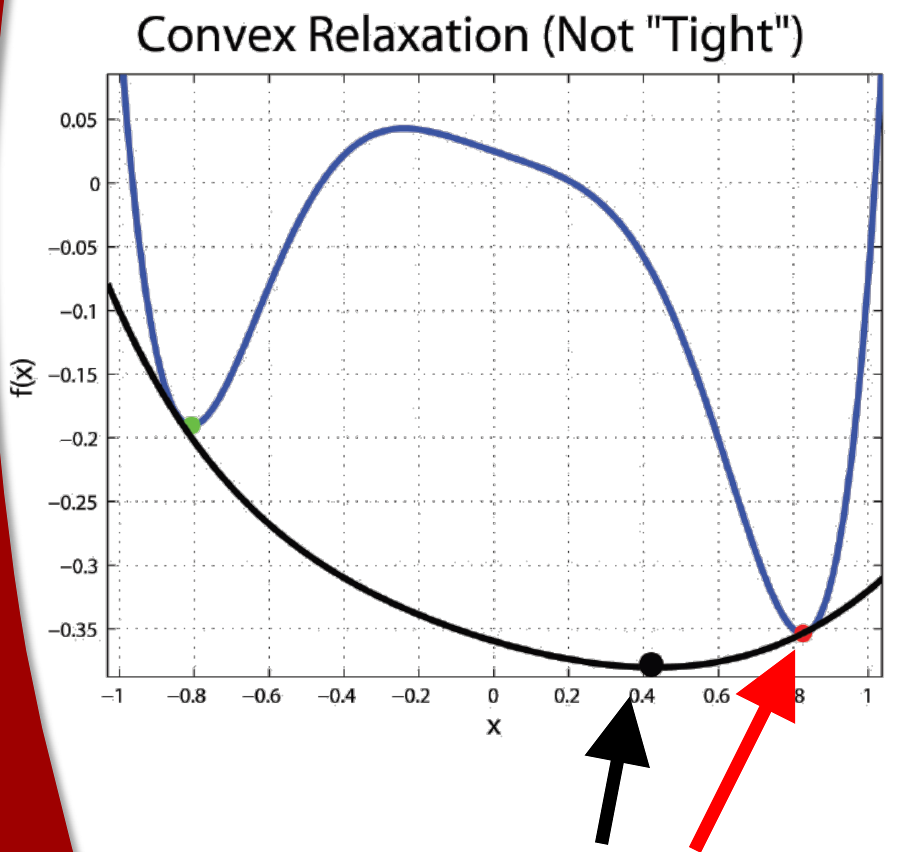
$\cancel{\mathbf{W} = xx^T}$ $\quad \mathbf{W} \succeq 0$ Semidefinite relaxation

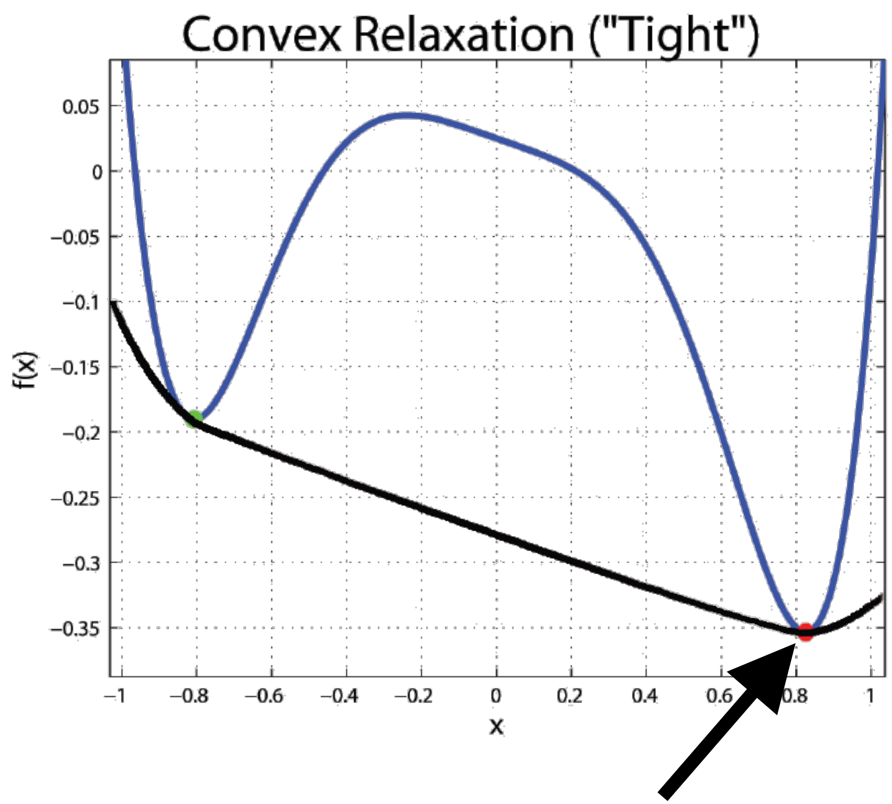
$$x = \begin{bmatrix} V_{d1} & V_{d2} & \ldots & V_{dn} & V_{q1} & V_{q2} & \ldots & V_{qn} \end{bmatrix}^T$$

Duality Gap

- A *physically meaningful* solution has "zero duality gap"
  - Same optimal objective value for classical OPF and semidefinite relaxation
  - **W** is rank one (subject to angle reference)
  - Optimal voltage profile recoverable from semidefinite relaxation
- The semidefinite relaxation may not give physically meaningful solutions
  - A gap between optimal objective values for classical OPF and semidefinite relaxation
  - **W** is not rank one (subject to angle reference)

Duality Gap Illustration





Relaxation does not find global optimum (non-zero duality gap)

Relaxation finds global optimum (zero duality gap)

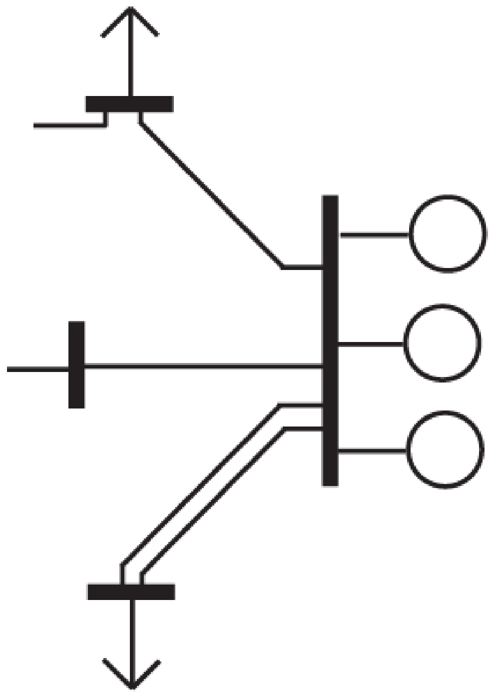
Techniques for Large-Scale OPF Problems

Overview

- Modeling advances
- Computational improvements
- Sufficient condition for global optimality

Modeling Advances

- Multiple generators at the same bus
  - Existing formulations limit total power injections at a bus
  - Analogy to consistent locational marginal price
- Flow limits on parallel lines and transformers
  - Existing formulations limit total flow between two buses
  - Limit flows on individual lines, including off-nominal voltage ratios and non-zero phase shifts

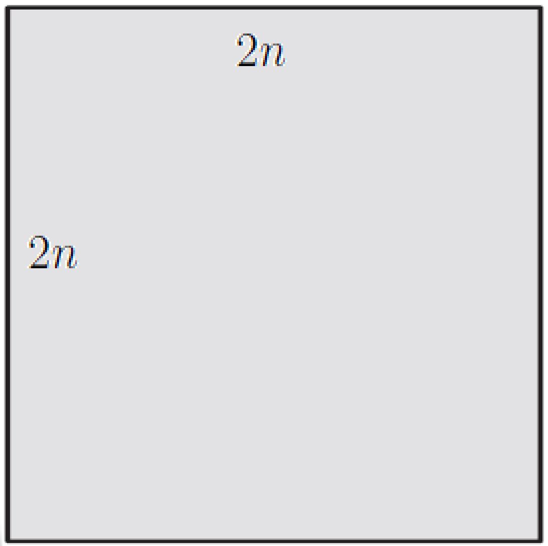


Matrix Completion Decomposition

- Use sparsity to reduce solver time
- Computational bottleneck is constraint $\mathbf{W} \succeq 0$
  - Scales as $(2n)^2$ for an *n*-bus system
- Replace with positive semidefinite constraints on many smaller matrices [Jabr '11]
  - Requires equality constraints between equivalent terms in different matrices

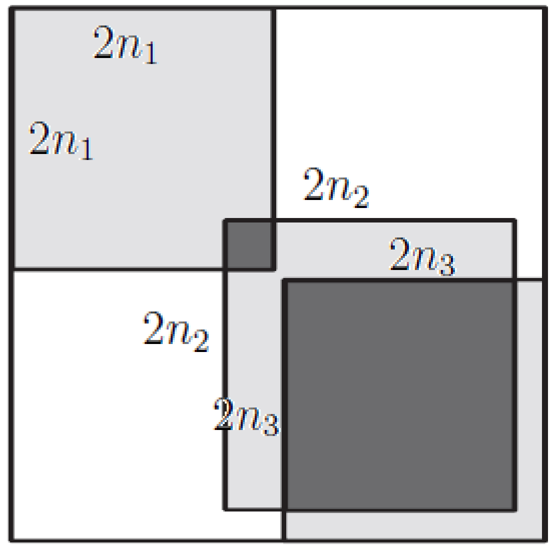
Decomposition Improvement

1. No decomposition



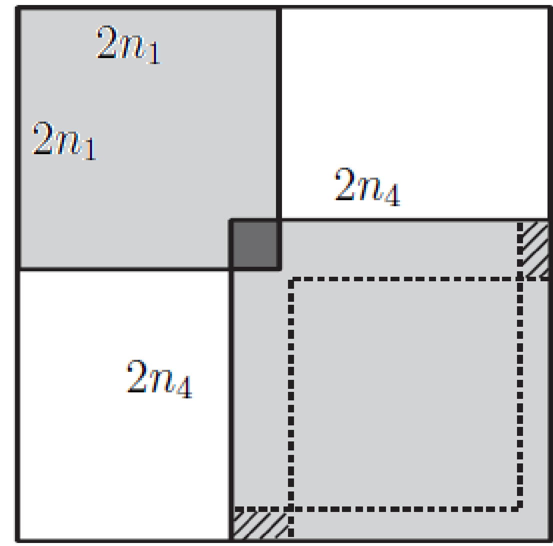
$2n \times 2n$

2. Decomposition in existing literature



$$\begin{aligned} & 2n_1 \times 2n_1 \\ + \; & 2n_2 \times 2n_2 \\ + \; & 2n_3 \times 2n_3 \\ + \; & \text{linking constraints (gray)} \end{aligned}$$

3. Proposed decomposition



$$\begin{aligned} & 2n_1 \times 2n_1 \\ + \; & 2n_4 \times 2n_4 \\ + \; & \text{linking constraints (gray)} \end{aligned}$$

Computational Results

- Matrix combination heuristic provides a factor of 2 to 3 speed improvement over existing decompositions

System	$2n \times 2n$	No Combining	Combining Heuristic	Speed Up Factor
IEEE 118-bus	6.63	4.84	2.06	2.349
IEEE 300-bus	69.45	13.18	5.71	2.309
Polish 2736-bus	–	2371.7	792.7	2.992
Polish 3012-bus	–	3578.5	1197.4	2.989

Solver Times (sec)

Sufficient Condition for Global Optimality

- Determine if the solution from a traditional solver is globally optimal
  - Use the Karush-Kuhn-Tucker conditions for optimality of the semidefinite program
    - Complementarity: $\mathrm{trace}\left(\mathbf{AW}\right) = 0$
    - Feasibility: $\mathbf{A} \succeq 0$
- Globally optimal solutions to many small test systems, indeterminate for some large systems

MATPOWER Implementation

- Proposed methods implemented in MATLAB code that integrates with MATPOWER
- Preparing code for public release

```
Solution satisfies rank and consistency conditions, 7343.05 seconds.
mineigratio = 1.5662e+005, zero_eval = 3.6567e-012
Objective Function Value = 1308578.12 $/hr
================================================================================
|     System Summary                                                           |
================================================================================

How many?                How much?              P (MW)            Q (MVAr)
---------------------    -------------------  -------------  -----------------
Buses              2736  Total Gen Capacity   28880.0        -2844.7 to 18056.1
Generators          420  On-line Capacity     20246.7        -1830.7 to 11450.0
Committed Gens      270  Generation (actual)  18394.7             2215.5
Loads              2048  Load                 18074.5             5339.5
  Fixed            2048    Fixed              18074.5             5339.5
  Dispatchable        0    Dispatchable          -0.0 of -0.0       -0.0
Shunts                1  Shunt (inj)             -0.0             -123.9
Branches           3504  Losses (I^2 * Z)       317.00            2162.10
Transformers        174  Branch Charging (inj)     -              5427.1
Inter-ties           17  Total Inter-tie Flow  2006.3              364.8
Areas                 4

                          Minimum                      Maximum
                 -------------------------  --------------------------------
Voltage Magnitude   0.994 p.u. @ bus 2164      1.120 p.u. @ bus 2320
Voltage Angle     -30.93 deg   @ bus 2190      4.23 deg   @ bus 246
P Losses (I^2*R)             -                 8.41 MW    @ line 28-25
Q Losses (I^2*X)             -               100.11 MVAr  @ line 28-25
Lambda P           94.43 $/MWh @ bus 2730    124.80 $/MWh @ bus 2321
Lambda Q           -4.85 $/MWh @ bus 2320     13.10 $/MWh @ bus 2164

================================================================================
|     Bus Data                                                                 |
================================================================================
 Bus      Voltage          Generation             Load          Lambda($/MVA-hr)
  #   Mag(pu) Ang(deg)   P (MW)   Q (MVAr)   P (MW)   Q (MVAr)     P        Q
----- ------- --------  --------  --------  --------  --------  -------  -------
    1  1.101    0.823       -         -         -         -      97.926   0.173
    2  1.106    2.543       -         -         -         -      97.542     -
    3  1.084  -22.568       -         -         -         -     110.962   0.125
```

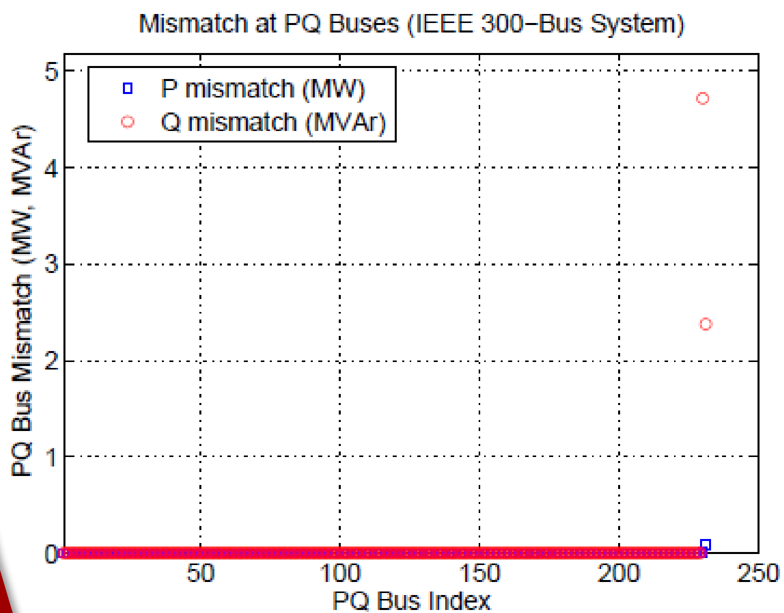
Results for Large-Scale Systems

Test System Results

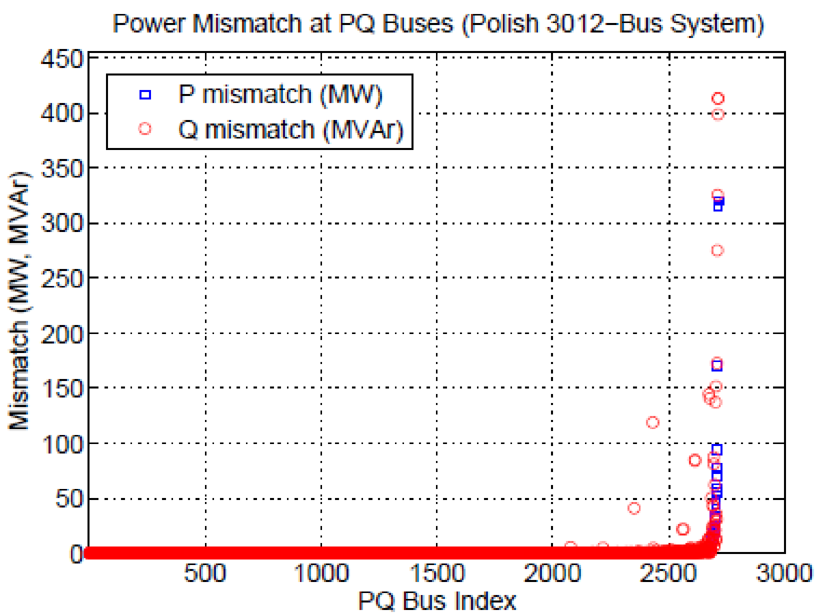
- Many problems have zero duality gap solutions
  - IEEE 14, 30, 57, and 118-bus test systems
  - Polish 2736, 2737, and 2746-bus systems in MATPOWER distribution
- Both small and large example systems with non-zero duality gap solutions

Large Systems Results

- Some solutions have zero duality gap
- Others have non-zero duality gap, but mismatch (using closest rank one **W** matrix) at only a few buses



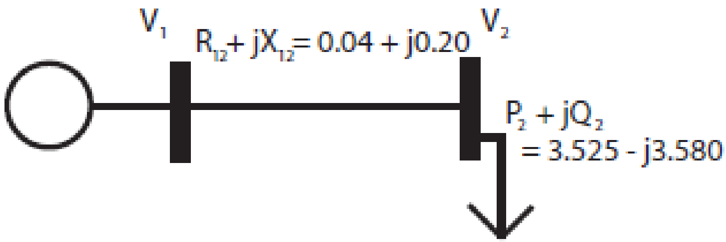
(a) Power Mismatch for IEEE 300-Bus System

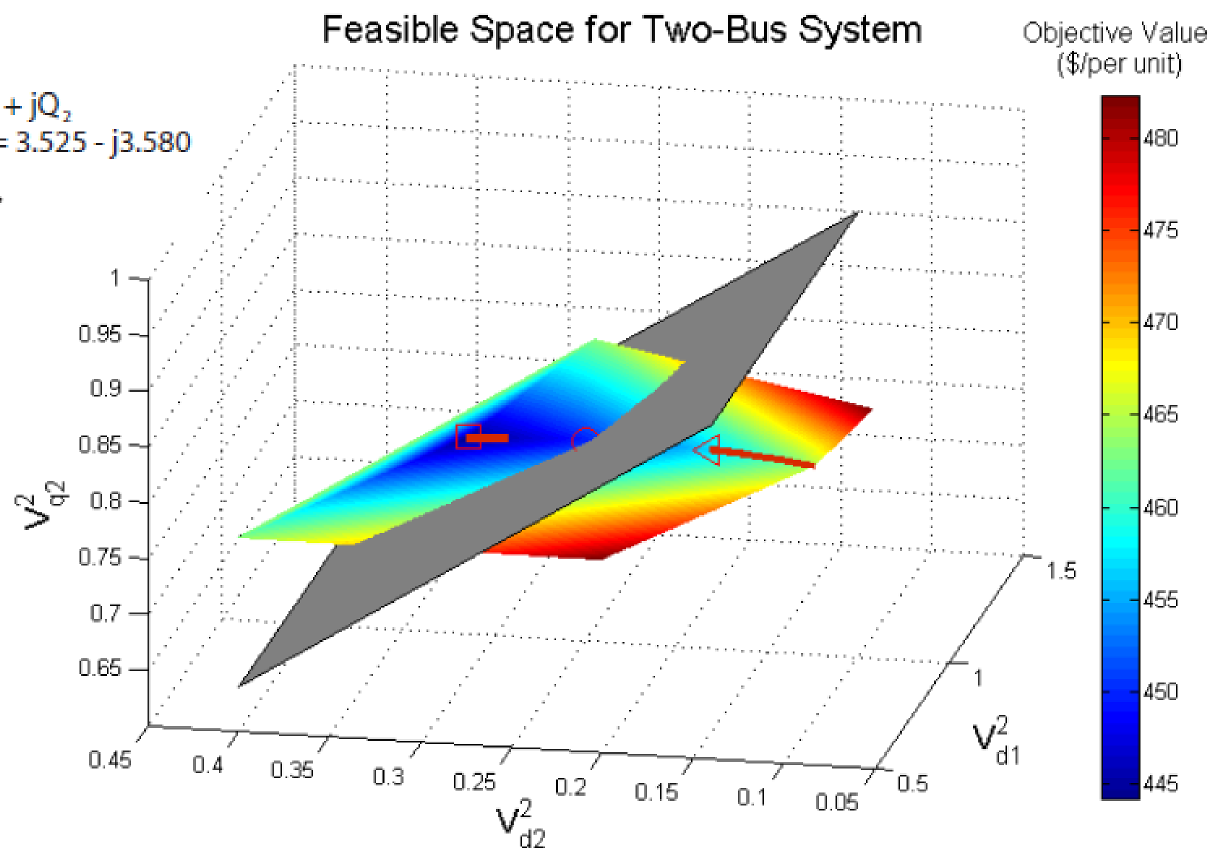


(b) Power Mismatch for Polish 3012-Bus System

Disconnected Feasible Space

- Two-bus example system [Bukhsh '11]





Conclusion

- A semidefinite relaxation finds a global optimum of many OPF problems
- Large-scale solver exploits power system sparsity using matrix decomposition
- Power injection mismatches in large systems appear isolated to small subsets of the network
- Illustration of non-convexities associated with non-zero duality gap solutions
- Sufficient condition test for global optimality of a candidate OPF solution

Related Publications

[1] B.C. Lesieutre, D.K. Molzahn, A.R. Borden, and C.L. DeMarco, “Examining the Limits of the Application of Semidefinite Programming to Power Flow Problems,” 4*9th Annual Allerton Conference on Communication, Control, and Computing (Allerton), 2011*, pp.1492-1499, 28-30 Sept. 2011.

[2] D.K. Molzahn, J.T. Holzer, and B.C. Lesieutre, and C.L. DeMarco, “Implementation of a Large-Scale Optimal Power Flow Solver Based on Semidefinite Programming,” To appear in *IEEE Transactions on Power Systems*.

[3] D.K. Molzahn, B.C. Lesieutre, and C.L. DeMarco, “An Approximate Method for Modeling ZIP Loads in a Semidefinite Relaxation of the OPF Problem," In preparation for submission to *IEEE Transactions on Power Systems, Letters.*

[4] D.K. Molzahn, B.C. Lesieutre, and C.L. DeMarco, "A Sufficient Condition for Global Optimality of Solutions to the Optimal Power Flow Problem," Submitted to *IEEE Transactions on Power Systems, Letters.*

[5] D.K. Molzahn, B.C. Lesieutre, and C.L. DeMarco, “A Sufficient Condition for Power Flow Insolvability with Applications to Voltage Stability Margins,” To appear in *IEEE Transactions on Power Systems*.

[6] D.K. Molzahn, V. Dawar, B.C. Lesieutre, and C.L. DeMarco, “Sufficient Conditions for Power Flow Insolvability Considering Reactive Power Limited Generators with Applications to Voltage Stability Margins,” To appear in *Bulk Power System Dynamics and Control - IX. Optimization, Security and Control of the Emerging Power Grid, 2013 IREP Symposium*, 25-30 Aug. 2013.

[7] D.K. Molzahn, B.C. Lesieutre, and C.L. DeMarco, “Investigation of Non-Zero Duality Gap Solutions to a Semidefinite Relaxation of the Power Flow Equations,” In preparation.

References

J. Lavaei and S. Low, "Zero Duality Gap in Optimal Power Flow Problem," *IEEE Transactions on Power Systems*, vol. 27, no. 1, pp. 92–107, February 2012.

R. Jabr, "Exploiting Sparsity in SDP Relaxations of the OPF Problem," *IEEE Transactions on Power Systems*, vol. 27, no. 2, pp. 1138–1139, May 2012.

W.A. Bukhsh, A. Grothey, K.I. McKinnon, and P.A. Trodden, "Local Solutions of Optimal Power Flow," University of Edinburgh School of Mathematics, Tech. Rep. ERGO 11-017, 2011.

B.C. Lesieutre and I.A. Hiskens, ""Convexity of the Set of Feasible Injections and Revenue Adequacy in FTR Markets," *IEEE Transactions on Power Systems*, vol. 20, no. 4, pp. 1790-1798, November 2005.

Questions?