Visualizing the Feasible Spaces of Challenging OPF Problems

Daniel Molzahn
Argonne National Laboratory

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Outline

- Overview of OPF feasible spaces and convex relaxations
- Example feasible spaces for both straightforward and challenging OPF problems
- Existing tools for exploring feasible spaces and their limitations
- A new algorithm for feasible space exploration
- Conclusions

Introduction and Background

Optimal Power Flow (OPF) Problem

- Optimization used to determine system operation
  - Minimize generation cost while satisfying physical laws and engineering constraints
  - Yields generator dispatches, line flows, etc.
- Large scale
  - Optimize dispatch for multiple states or countries
- Many related problems:
  - State estimation, unit commitment, transmission switching, contingency analysis, voltage stability margins, etc.

> "Today, 50 years after the problem was formulated, we still do not have a fast, robust solution technique for the full ACOPF."
> R.P. O'Neill, Chief Economic Advisor, US Federal Energy Regulatory Commission, 2013.

Feasible Spaces of OPF Problems

- Defined by the equality and inequality constraints
  - Equality constraints: power flow equations
  - Inequality constraints: engineering limitations
- Geometry of the feasible space is a key aspect of OPF problem difficulty
- Generally non-convex, may have multiple local minima and disconnected components

Classical OPF Problem

$$\min_{V_d, V_q} \quad \sum_{k \in \mathcal{G}} \left(c_{2k} P_{Gk}^2 + c_{1k} P_{Gk} + c_{0k} \right)$$

Generation Cost

subject to

$$P_{Gk}^{\min} \leq P_{Gk} \leq P_{Gk}^{\max}$$

$$Q_{Gk}^{\min} \leq Q_{Gk} \leq Q_{Gk}^{\max}$$

Engineering Constraints

$$\left(V_k^{\min}\right)^2 \leq V_{dk}^2 + V_{qk}^2 \leq \left(V_k^{\max}\right)^2$$

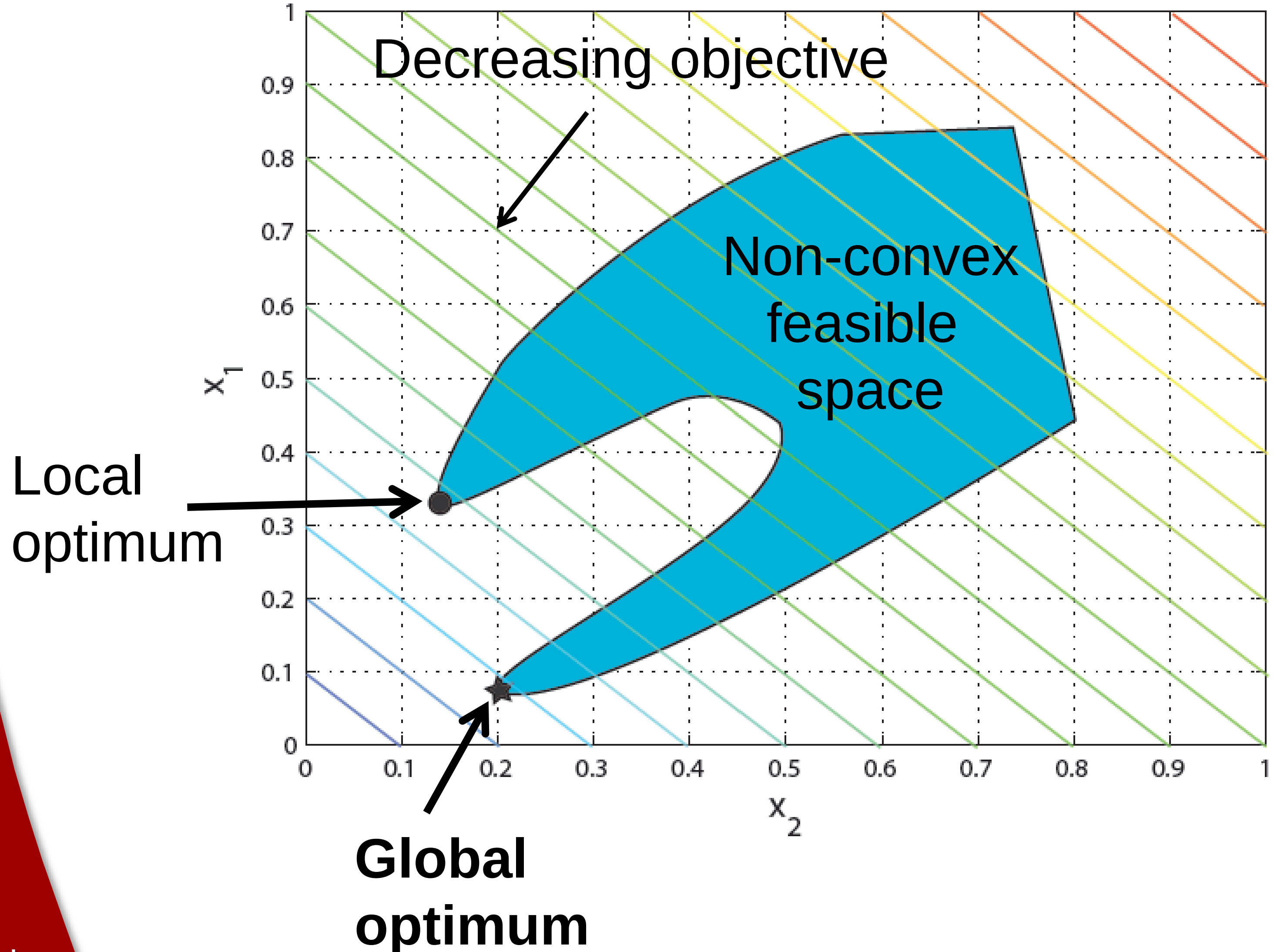
$$|S_{lm}| \leq S_{lm}^{\max}$$

Physical Laws

$$P_{Gk} - P_{Dk} = V_{dk} \sum_{i=1}^{n} \left(\mathrm{G}_{ik} V_{di} - \mathrm{B}_{ik} V_{qi} \right) + V_{qk} \sum_{i=1}^{n} \left(\mathrm{B}_{ik} V_{di} + \mathrm{G}_{ik} V_{qi} \right)$$

$$Q_{Gk} - Q_{Dk} = V_{dk} \sum_{i=1}^{n} \left(-\mathrm{B}_{ik} V_{di} - \mathrm{G}_{ik} V_{qi} \right) + V_{qk} \sum_{i=1}^{n} \left(\mathrm{G}_{ik} V_{di} - \mathrm{B}_{ik} V_{qi} \right)$$

Convex Relaxation



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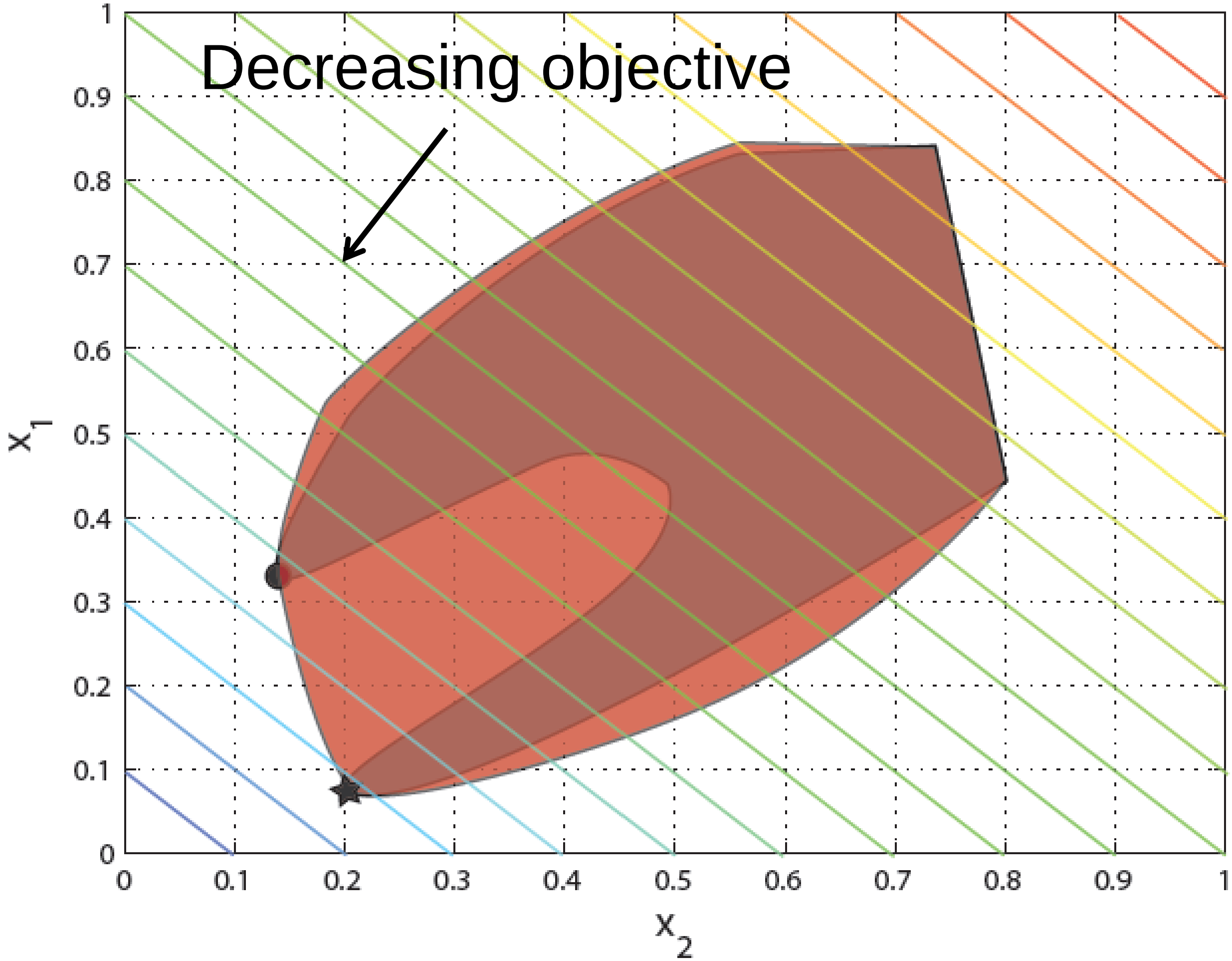
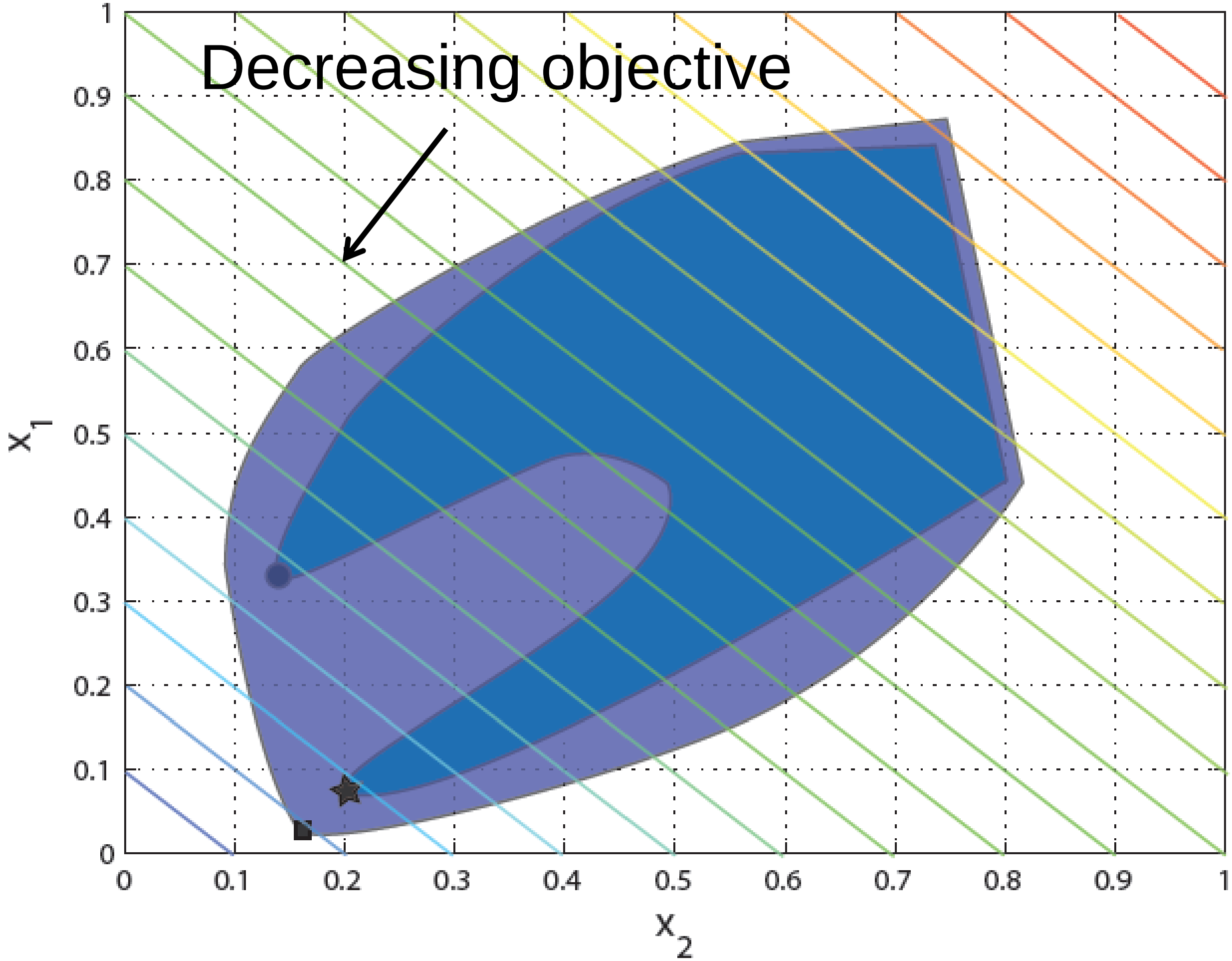


Relaxation finds global optimum
(**zero relaxation gap**)

Convex Relaxation



Relaxation does not find global optimum
(**non-zero relaxation gap**)

Semidefinite Programming

- Convex optimization
- Interior point methods solve for the global optimum in polynomial time

$$\min_{\mathbf{W}} \operatorname{trace}(\mathbf{BW})$$

$$\text{subject to}$$

$$\operatorname{trace}(\mathbf{A}_i\mathbf{W}) = c_i$$

$$\mathbf{W} \succeq 0$$

where $\mathbf{B}$ and $\mathbf{A}_i$ are specified symmetric matrices

Recall: $\operatorname{trace}(\mathbf{A}^{\mathsf{T}}\mathbf{W}) = \mathbf{A}_{11}\mathbf{W}_{11} + \mathbf{A}_{12}\mathbf{W}_{12} + \ldots + \mathbf{A}_{nn}\mathbf{W}_{nn}$

$$\mathbf{W} \succeq 0 \text{ if and only if } \operatorname{eig}(\mathbf{W}) \geq 0$$

Semidefinite Relaxations

- Write power flow equations as $x^{\intercal}\mathbf{A}_i x = c_i$
  where $x = \begin{bmatrix} V_{d1} & V_{d2} & \ldots & V_{dn} & V_{q1} & V_{q2} & \ldots & V_{qn} \end{bmatrix}^{\intercal}$

- Define matrix $\mathbf{W} = xx^{\intercal}$
- Rewrite as $\mathrm{trace}\left(\mathbf{A}_i\mathbf{W}\right) = c_i$ and $\begin{cases} \mathbf{W} \succeq 0 \\ \mathrm{rank}\left(\mathbf{W}\right) = 1 \end{cases}$

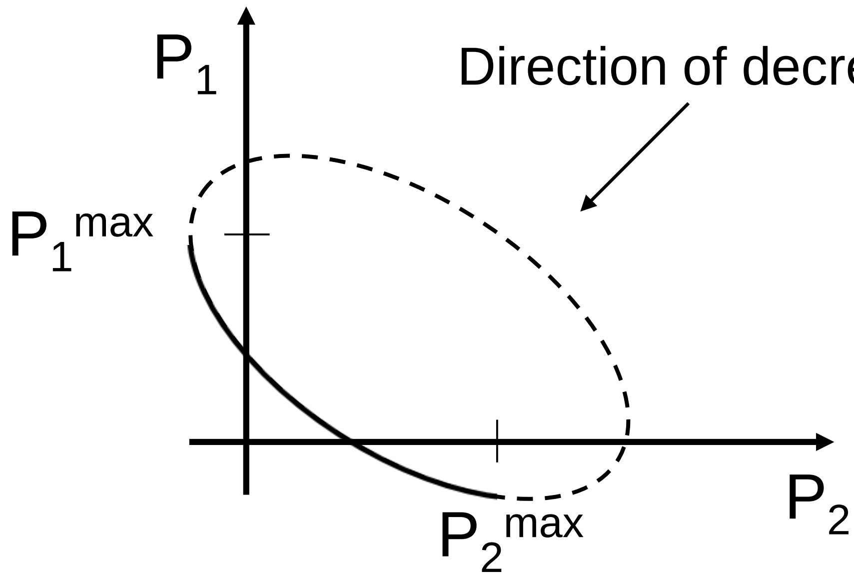
- Relaxation: do not enforce the rank constraint
  - $\mathrm{rank}\left(\mathbf{W}\right) = 1$ implies zero relaxation gap ("exact" solution) and recovery of the globally optimal voltage profile
    [Lavaei & Low '12]
  - Generalizable to hierarchies of convex relaxations
    [Lasserre '01, M. & Hiskens '14, M. & Hiskens '15, Josz & M., in review]

Example Feasible Spaces

Universally Convexifiable Feasible Spaces

- A tree network* has a feasible space with a Pareto front that is equivalent to the Pareto front of its convex hull [Zhang & Tse '11]

* (satisfying certain non-trivial conditions)



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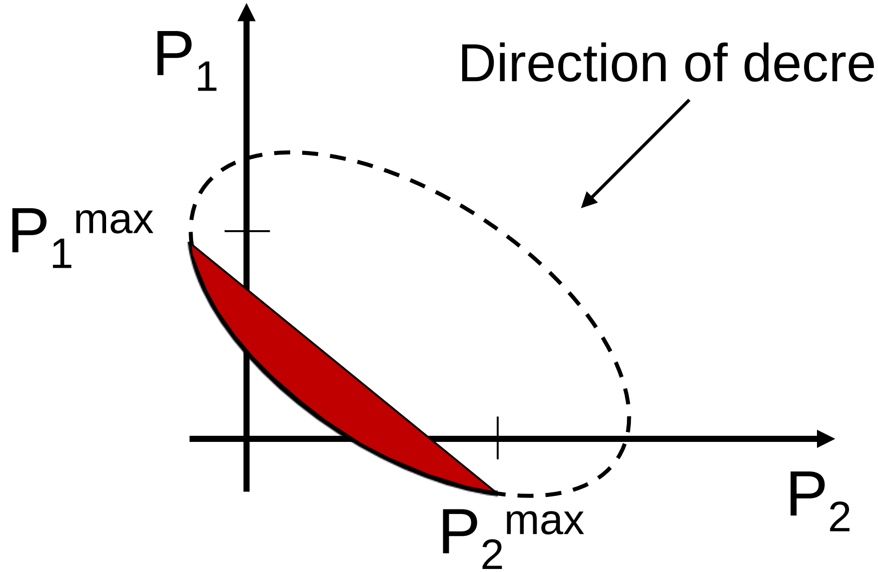
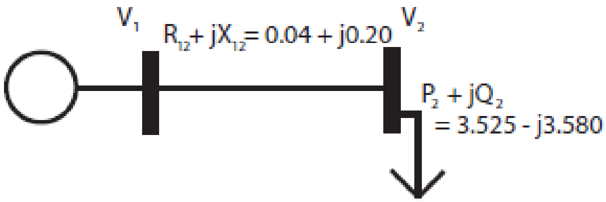
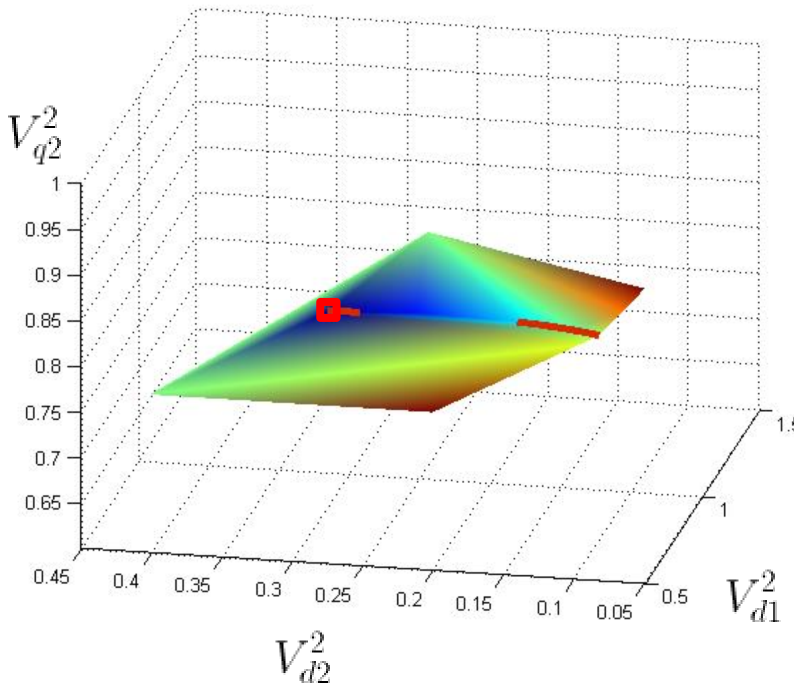


Disconnected Feasible Space

- Two-bus example OPF problem
  [Bukhsh et al. '11]



Feasible Space of the
Semidefinite Relaxation



$$\boxed{V_2^{max} = 1.05 \text{ per unit}}$$

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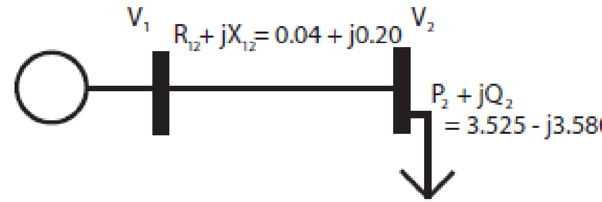

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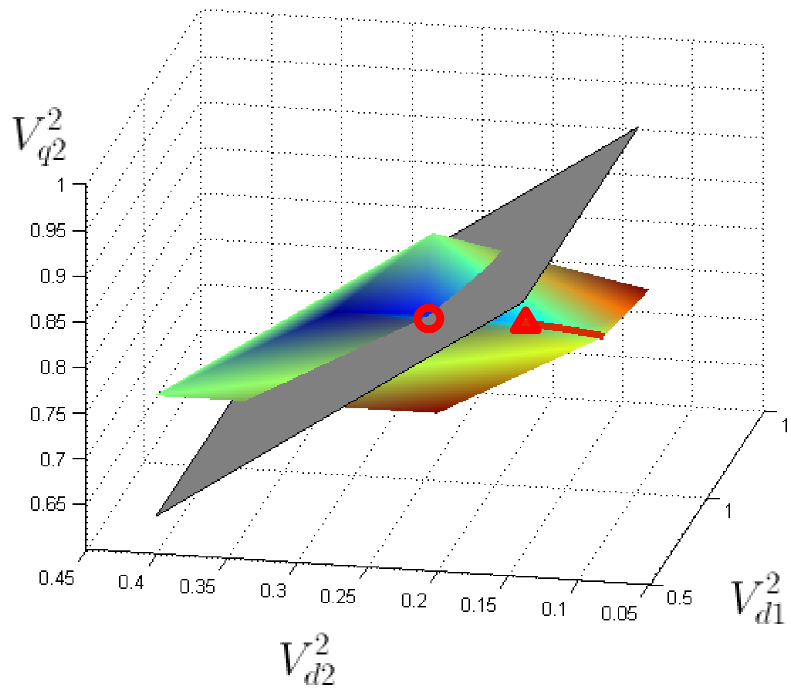
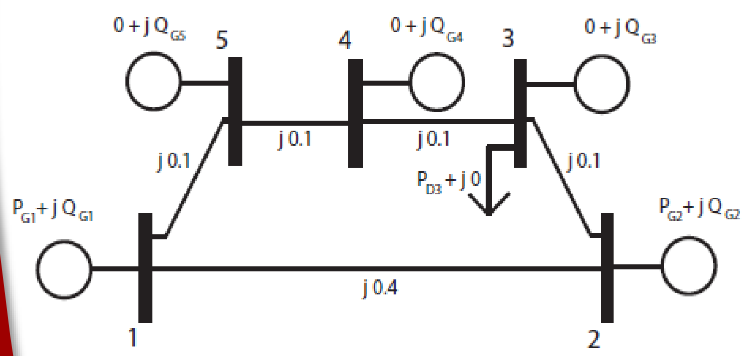


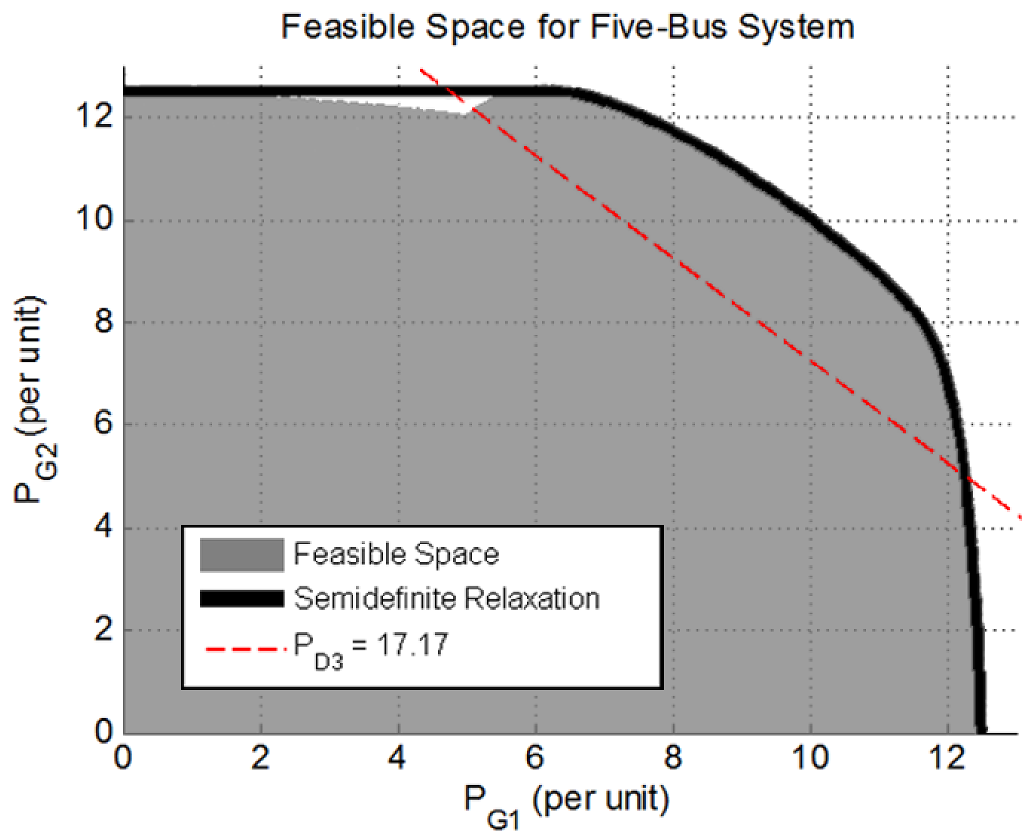
$$\boxed{V_2^{max} = 1.02 \text{ per unit}}$$

Examples

Non-Convex Space for a Lossless System

- Five-bus example OPF problem [Lesieutre & Hiskens '05]





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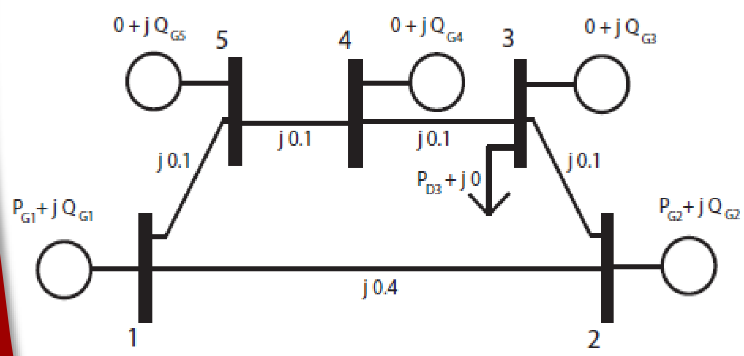

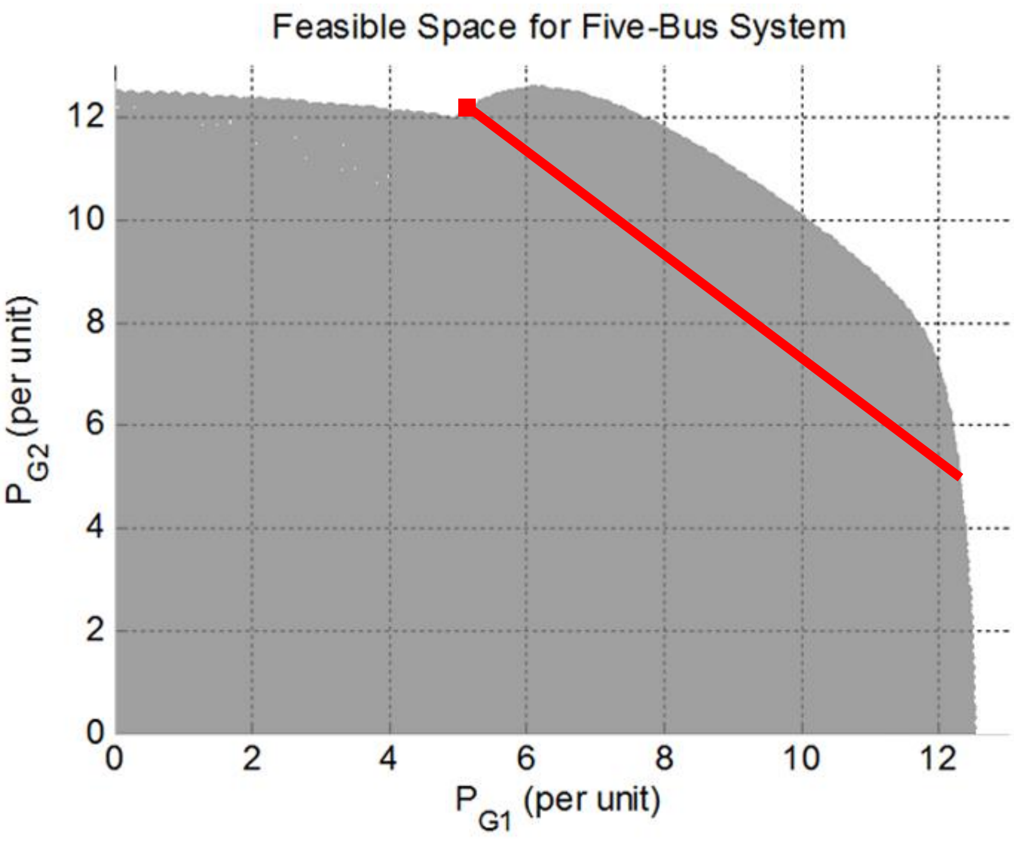

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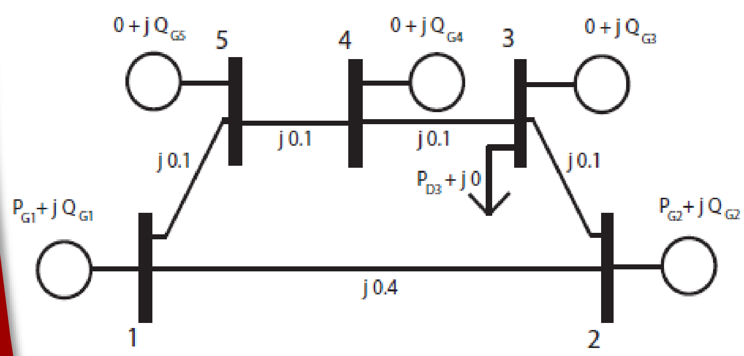

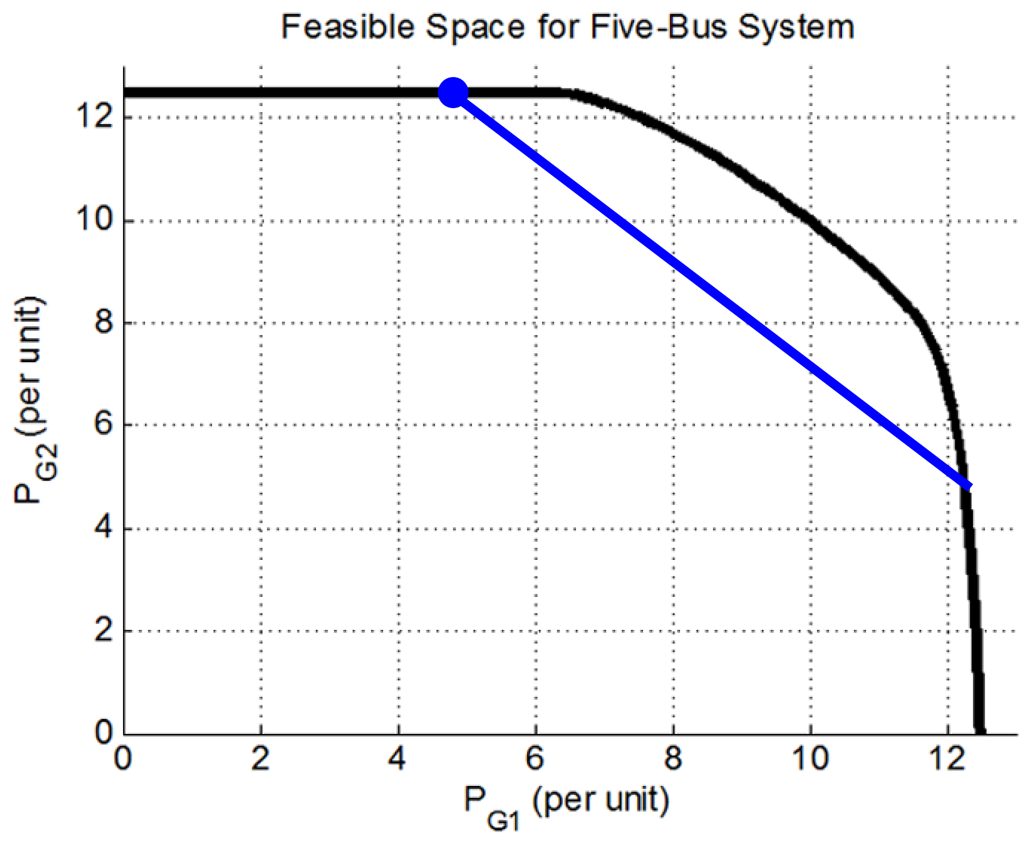

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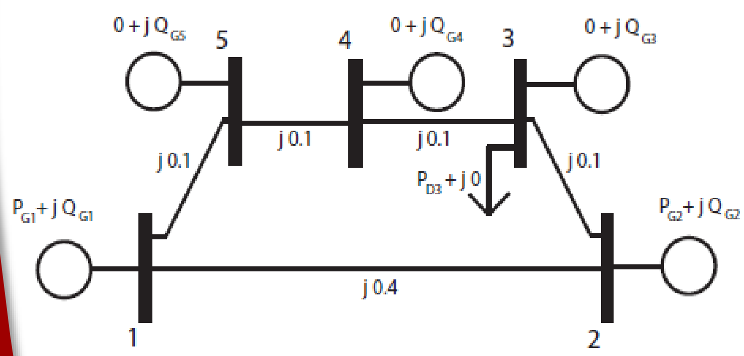
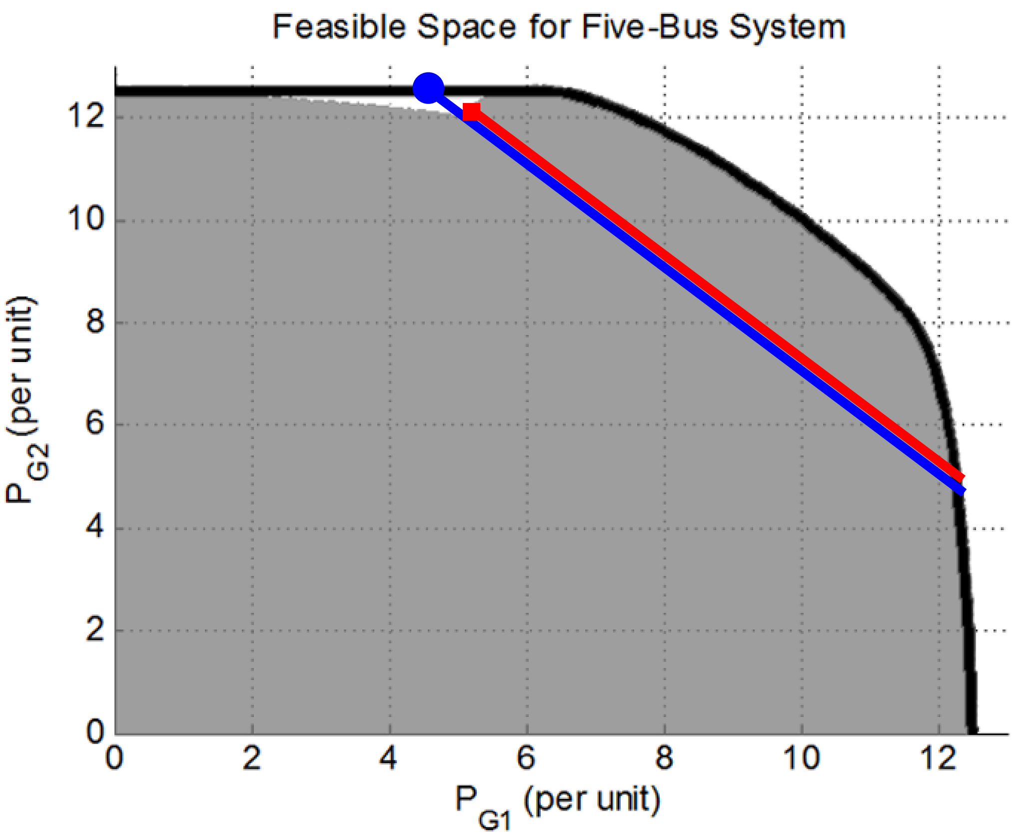
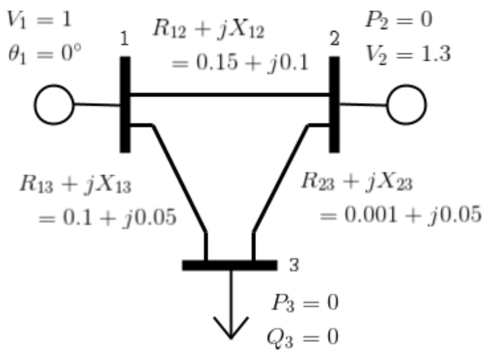


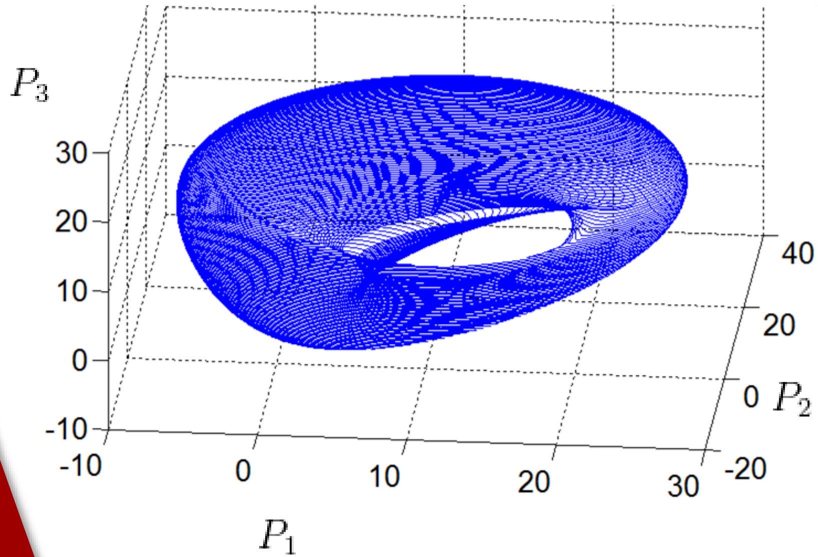


Hole in the Feasible Space

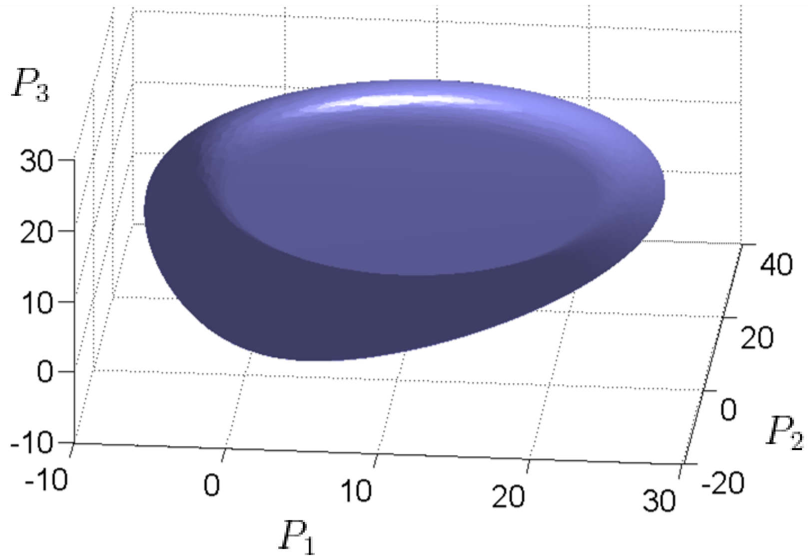
- Three-bus example OPF problem
  [M., Baghsorkhi & Hiskens '15]



Feasible Space of the OPF Problem



Feasible Space of the Semidefinite Relaxation



"Rules of Thumb" Associated with Challenging OPF Problems

- Systems where generators have limited ability to absorb reactive power
- "Low-voltage" power flow solutions within the admissible voltage range
- Tight limits on apparent power flows

Goal: Extend and formalize these "rules of thumb" and apply to large test cases.

This requires new computational tools to study small test cases.

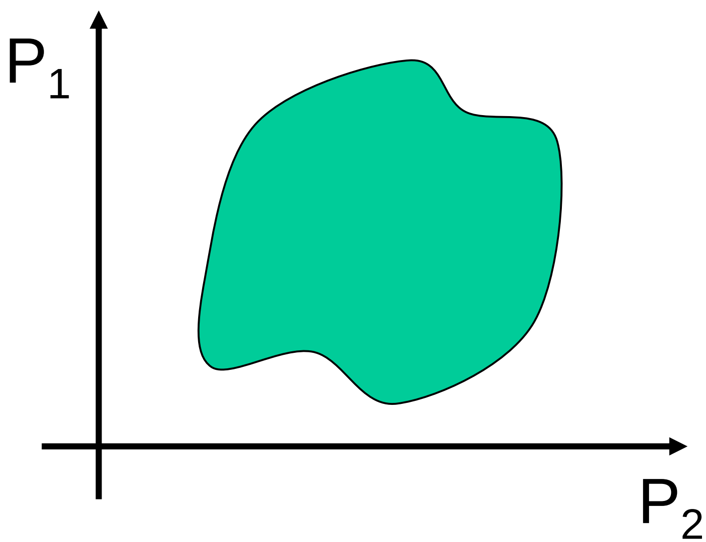
Existing Tools for Exploring OPF Feasible Spaces

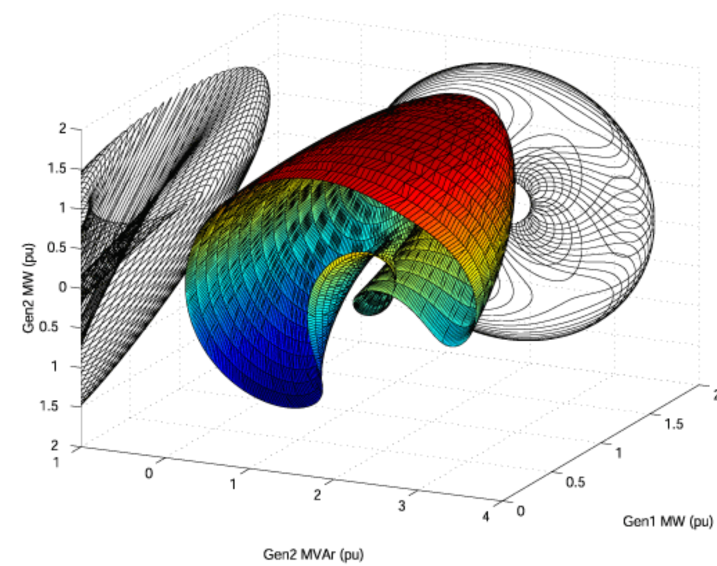
“One-Off” Approach to Previous Examples

- All previous examples were generated as “special cases” exploiting specific problem structure
  - 2-bus system: reduce to cubic equation, solve explicitly
  - 5-bus system: analytic expression that exploits problem specific symmetries
  - 3-bus system: uses a homotopy approach that is only suitable for very small problems

Continuation Along Power Flow Boundary

- Algorithm: [Hiskens & Davy '01]
  - Start at an interior point
  - Continuation method to reach the boundary
  - Continuation along contours of the boundary by enforcing singular power flow Jacobian





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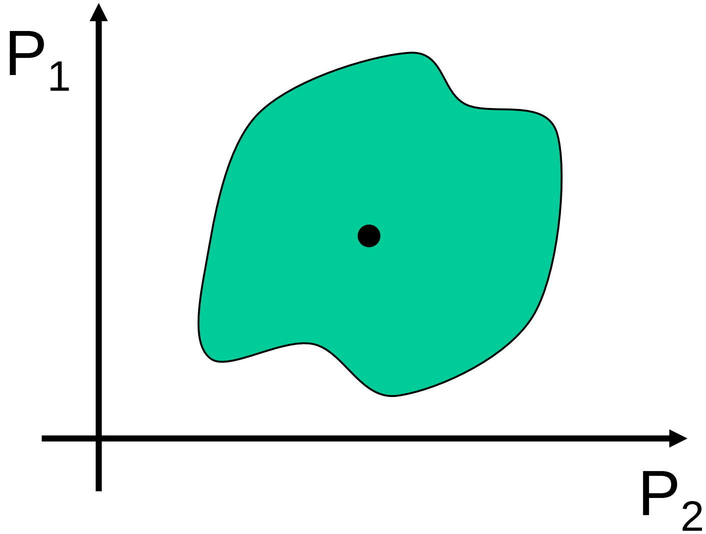

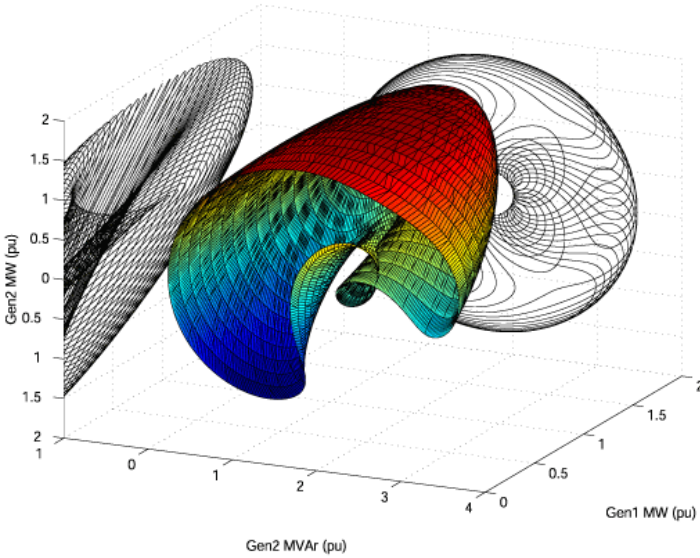

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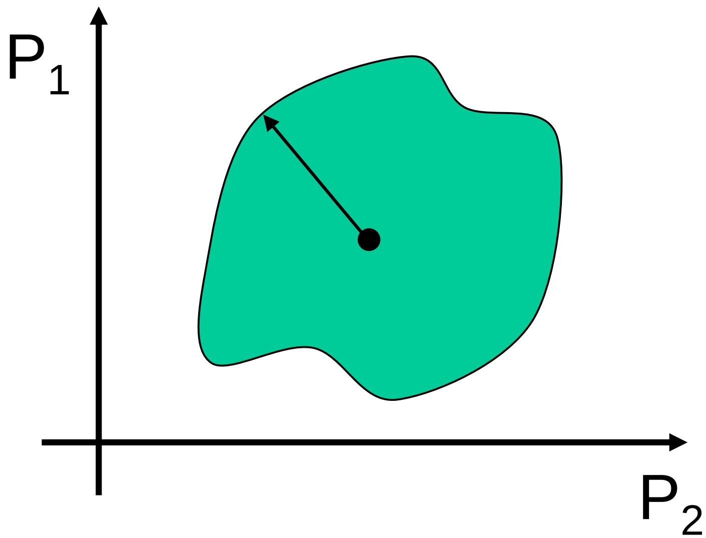

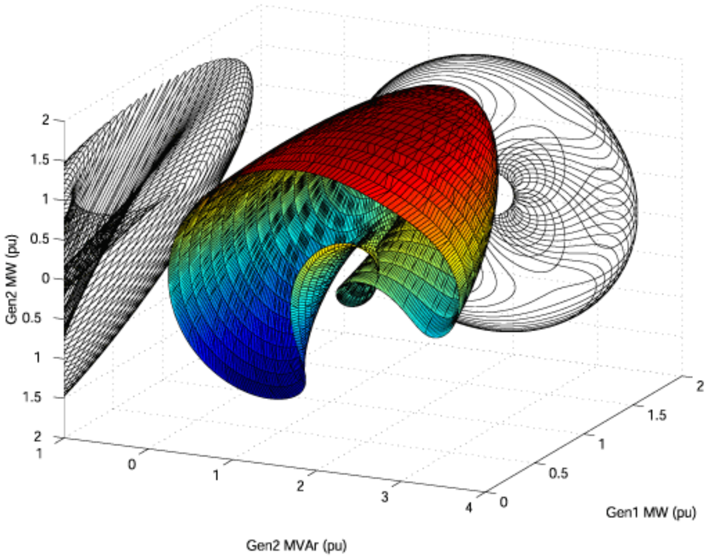

Existing Tools

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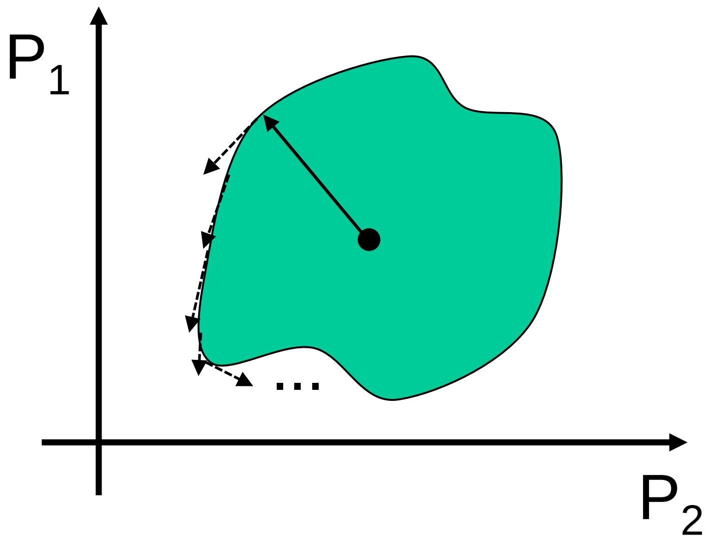
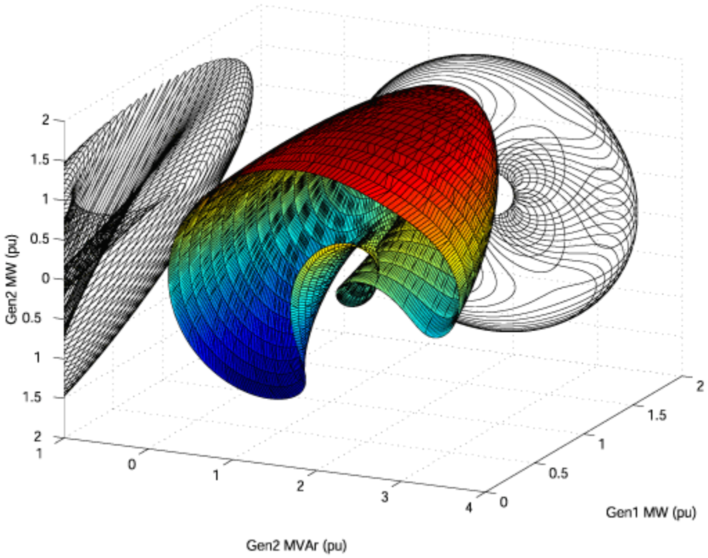




Limitations of Existing Approaches

- Approach by Hiskens & Davy not guaranteed to obtain entire feasible region
  - Need an initial interior point
  - Only finds a single connected component
  - May fail with sharp non-convexities
- Other approaches are only applicable to very small systems or systems with special symmetries

Studying difficult problems raises concerns regarding the possible failure of existing tools

New Algorithm for Visualizing OPF Feasible Spaces

Numerical Polynomial Homotopy Continuation (NPHC) Method

- Guaranteed to find all complex solutions to systems of polynomial equalities
- Limited to small ($\leq$10 bus) systems
  - Recent work may enable solution of somewhat larger problems

[M., Mehta, & Niemerg '16]

Homotopy from $t = 1 \rightarrow 0$:

Random complex scalar

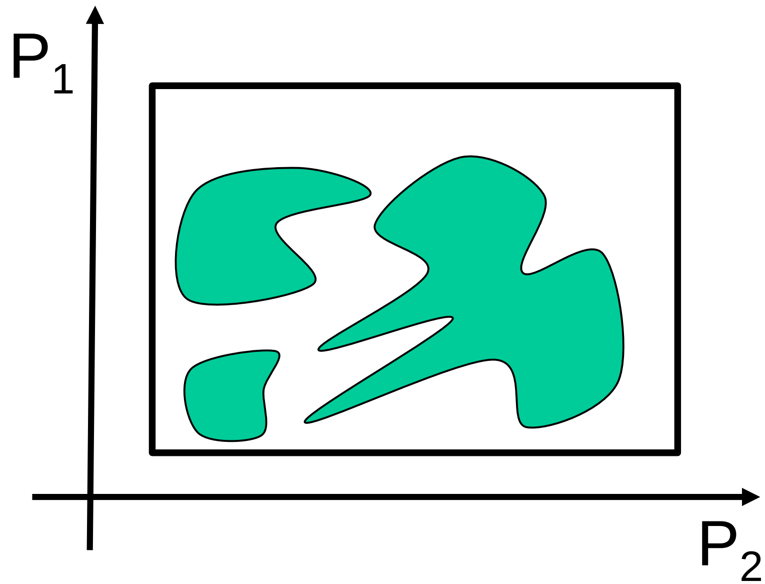
$$(1-t)\,f(x) + \kappa\,t\,g(x) = 0$$

Target polynomial system

Simple polynomial system with known solutions

Feasible Space Algorithm

1. Use convex relaxations to tighten the OPF constraints
2. Use "gridding" to convert from inequalities to equalities
3. Use convex relaxations to eliminate provably infeasible points
4. Calculate all power flow solutions at each grid point using the NPHC method
5. Select solutions that satisfy all constraints



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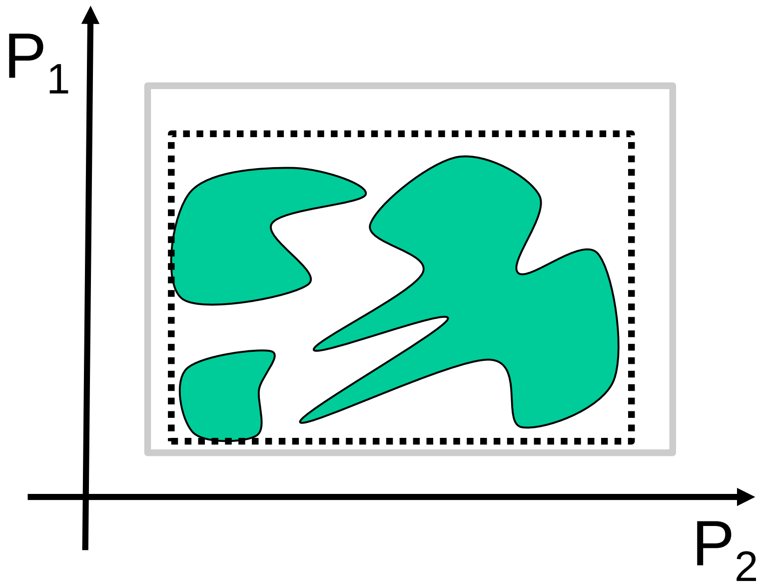

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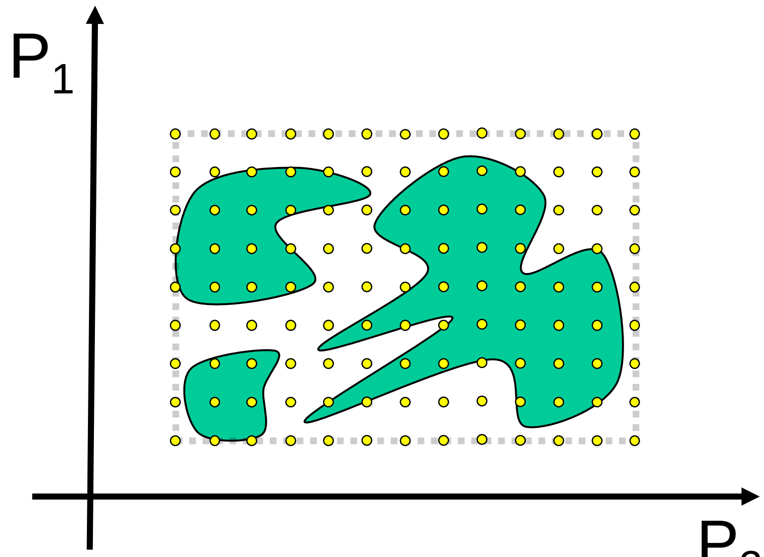

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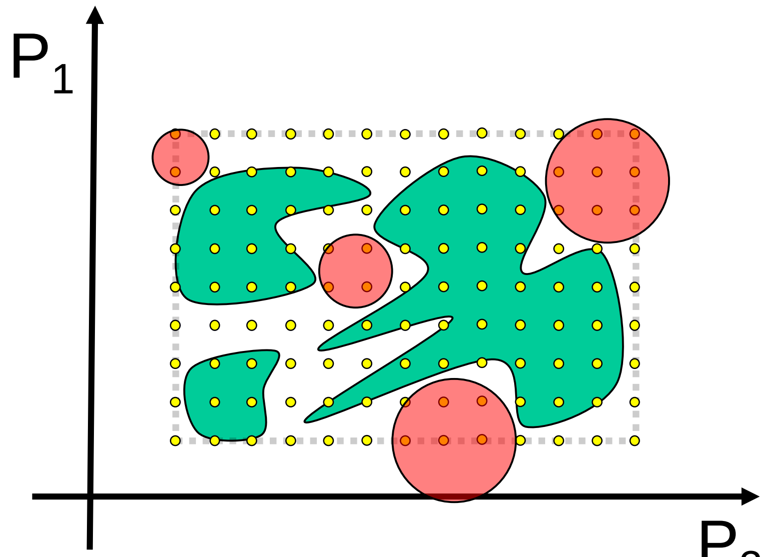

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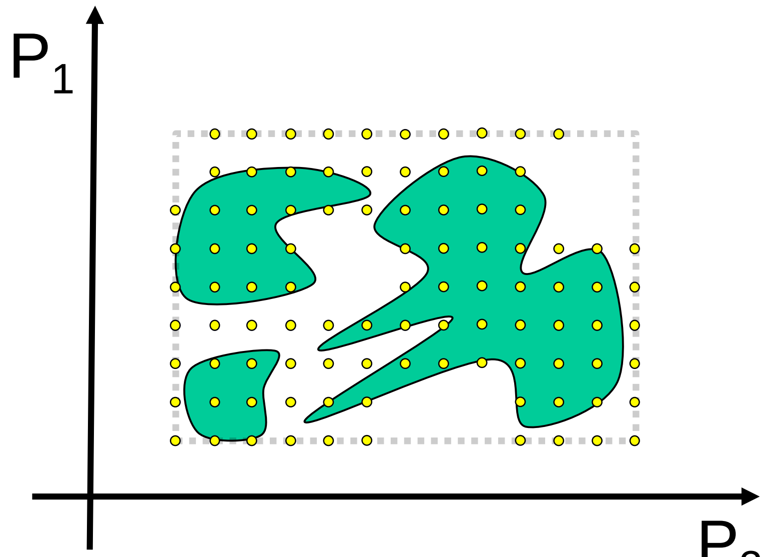

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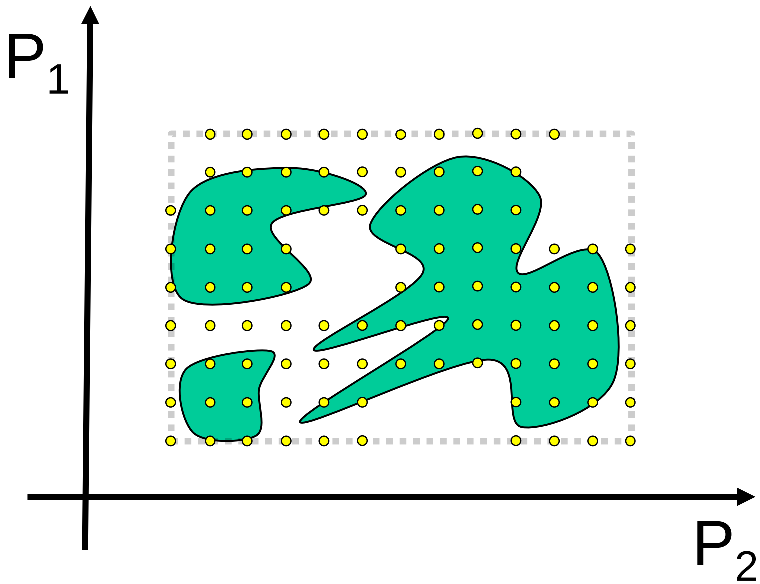

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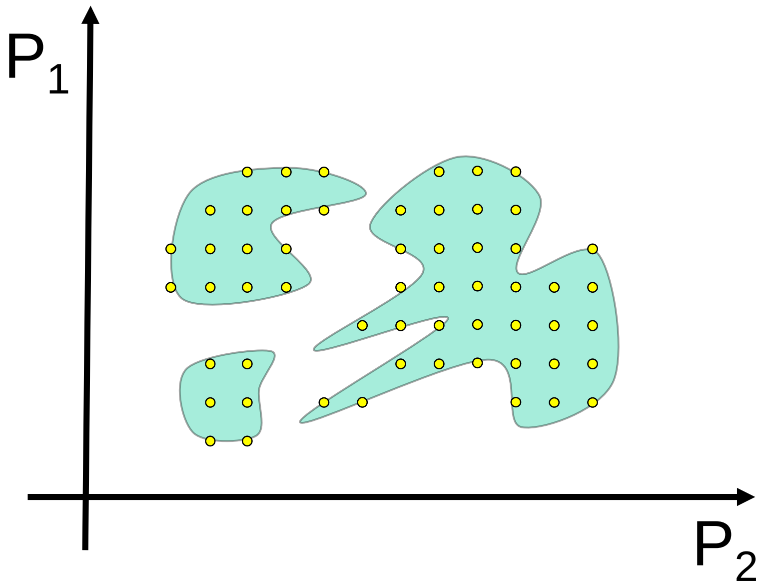
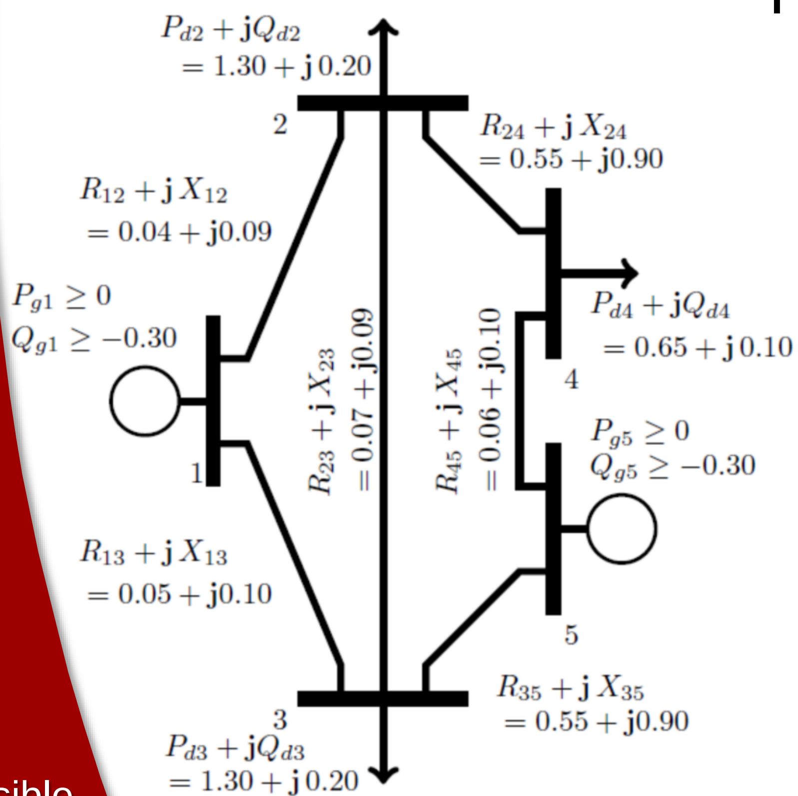


Advantages

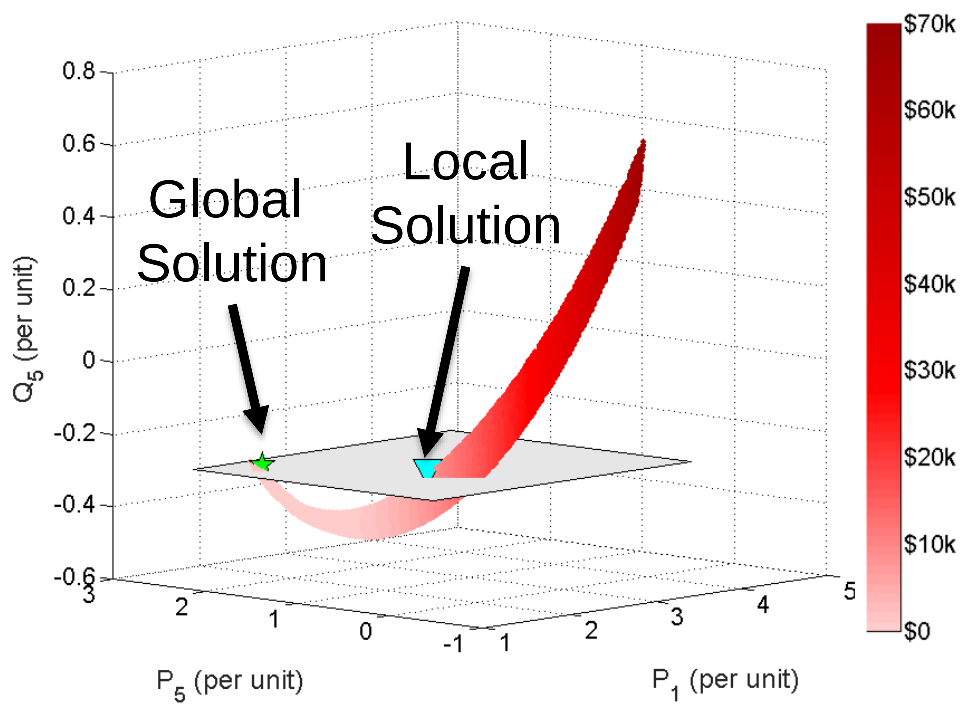
- Guaranteed to obtain the complete feasible space, within the discretization chosen for the grid
  - Inherits robustness of NPHC method
- Can hotstart NPHC method using solutions at a nearby grid point
- Applicable to many small test cases known to be challenging

Results: Five-Bus System

- Five-bus example OPF problem (modified objective) [Bukhsh et al. ‘13]



Feasible Space of OPF Problem



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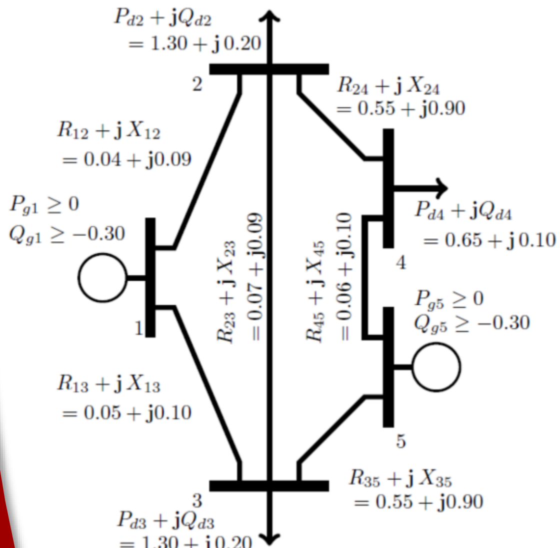

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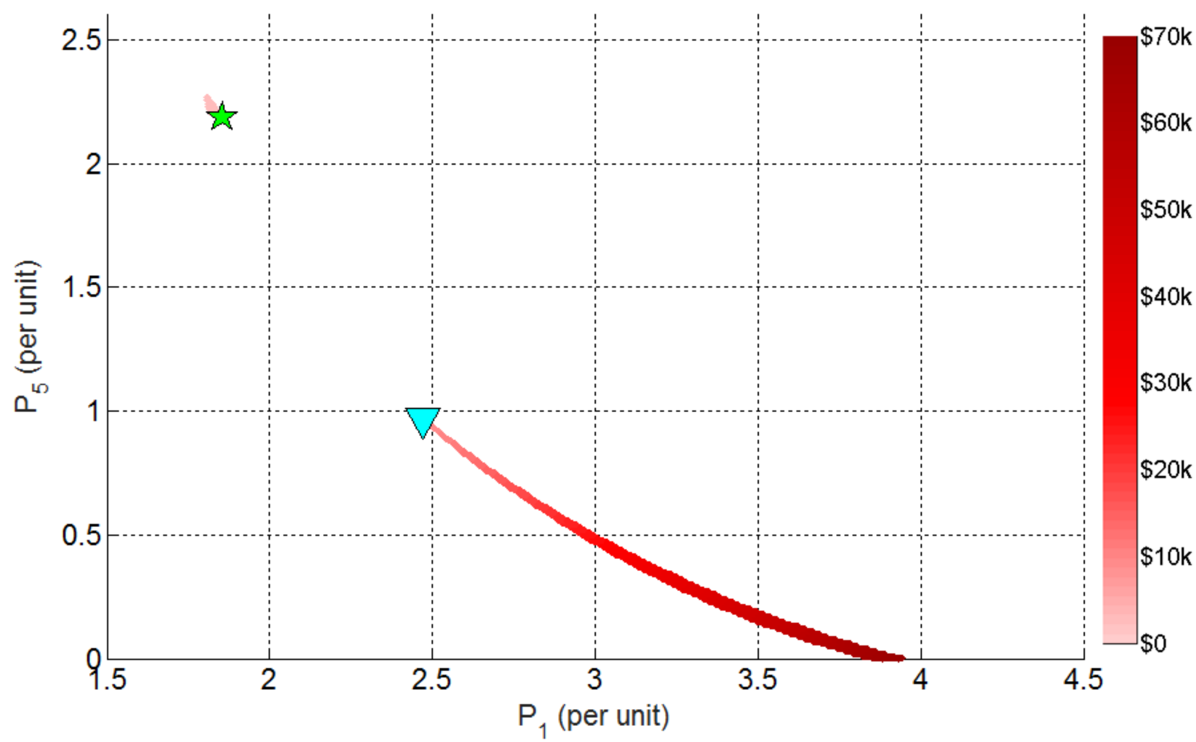

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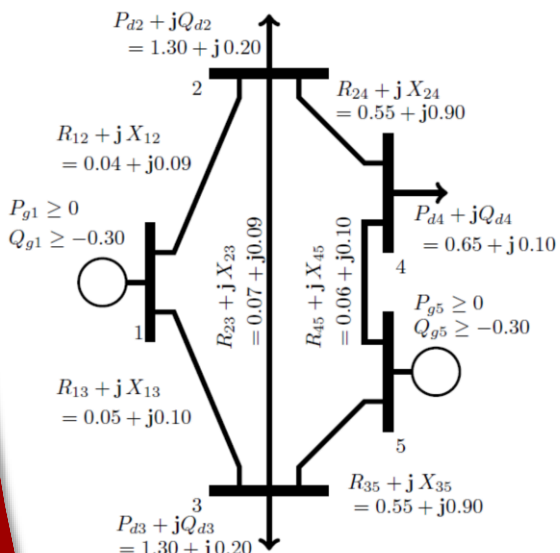
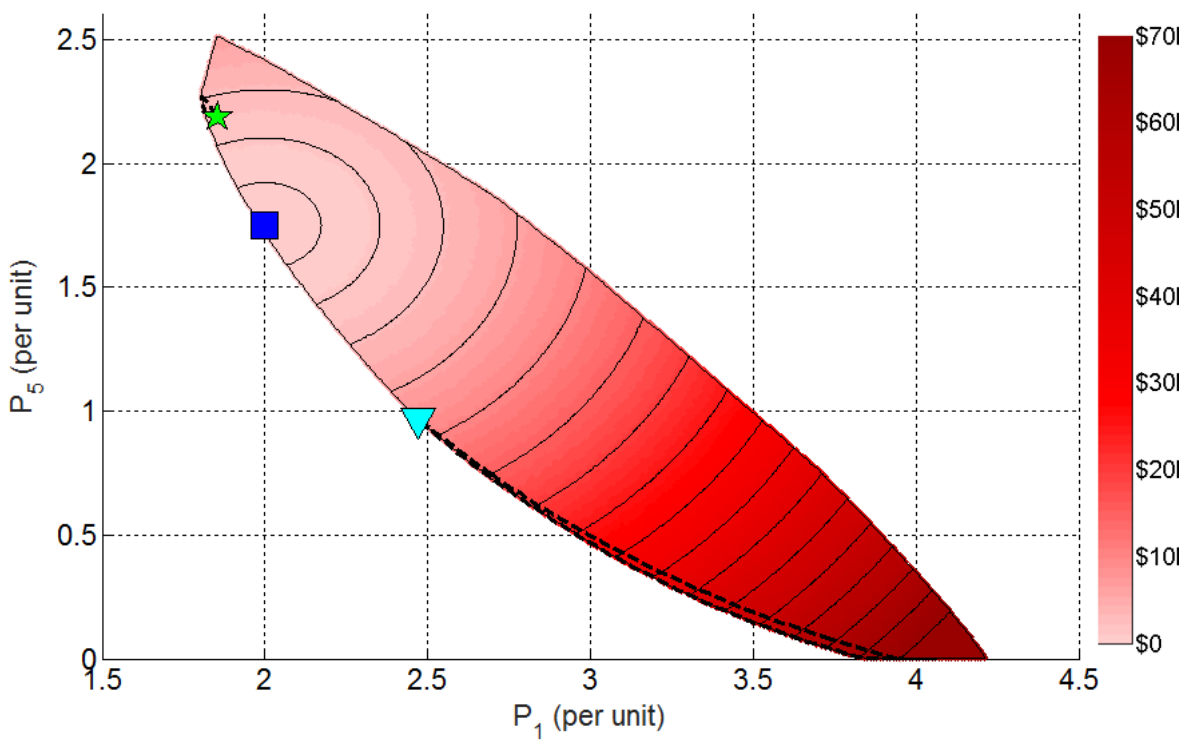


Feasible Space of SDP Relaxation



Feasible Space Algorithm

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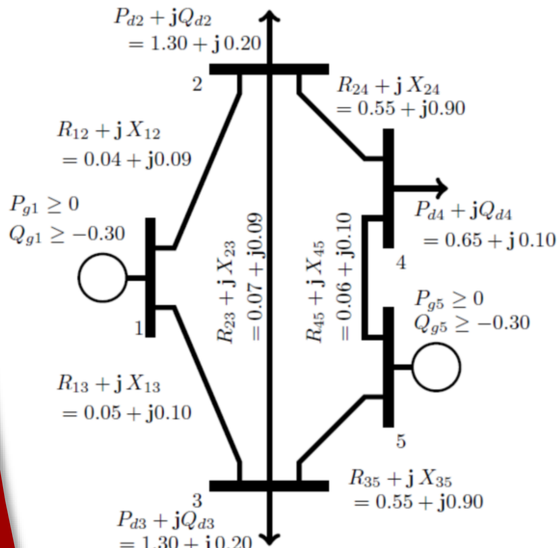
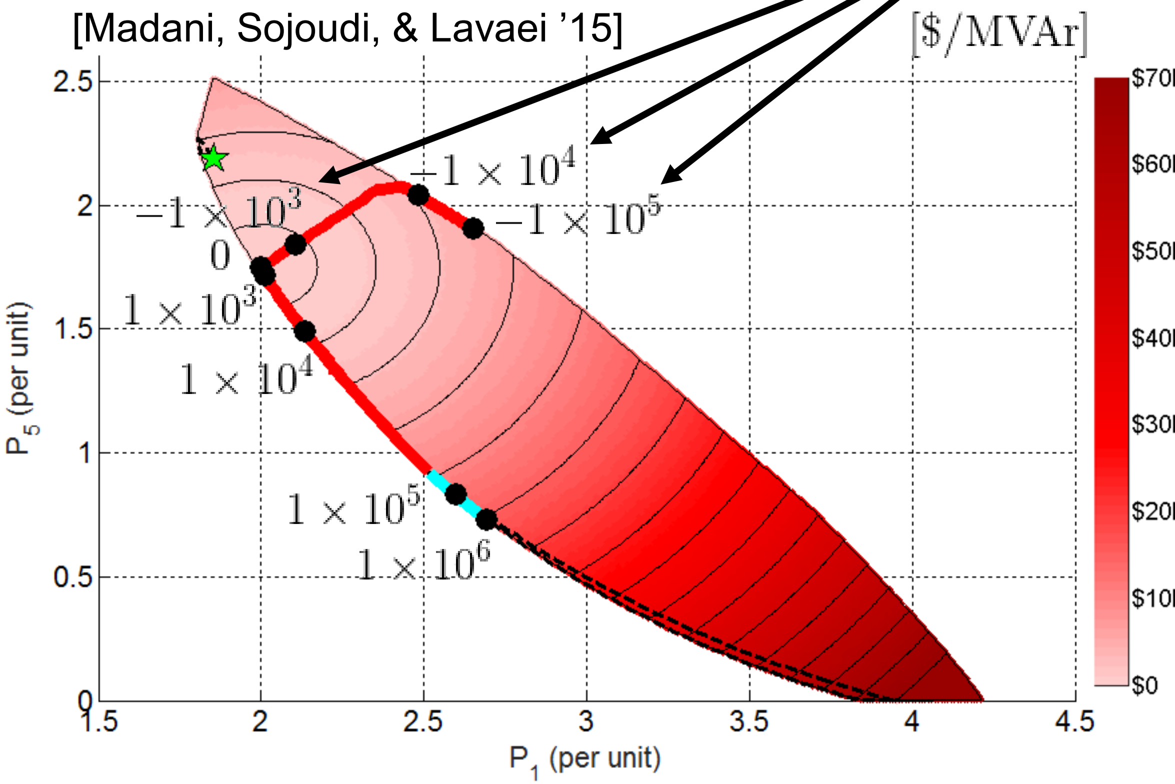
$$\min \sum_{k \in \mathcal{G}} \left(c_{2,k} P_{Gk}^2 + c_{1,k} P_{Gk} + c_{0k} \right) + \epsilon_b Q_{Gk}$$



[Madani, Sojoudi, & Lavaei '15]



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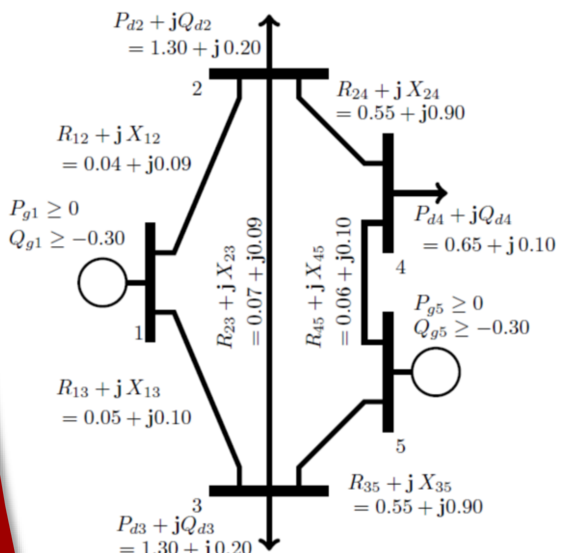
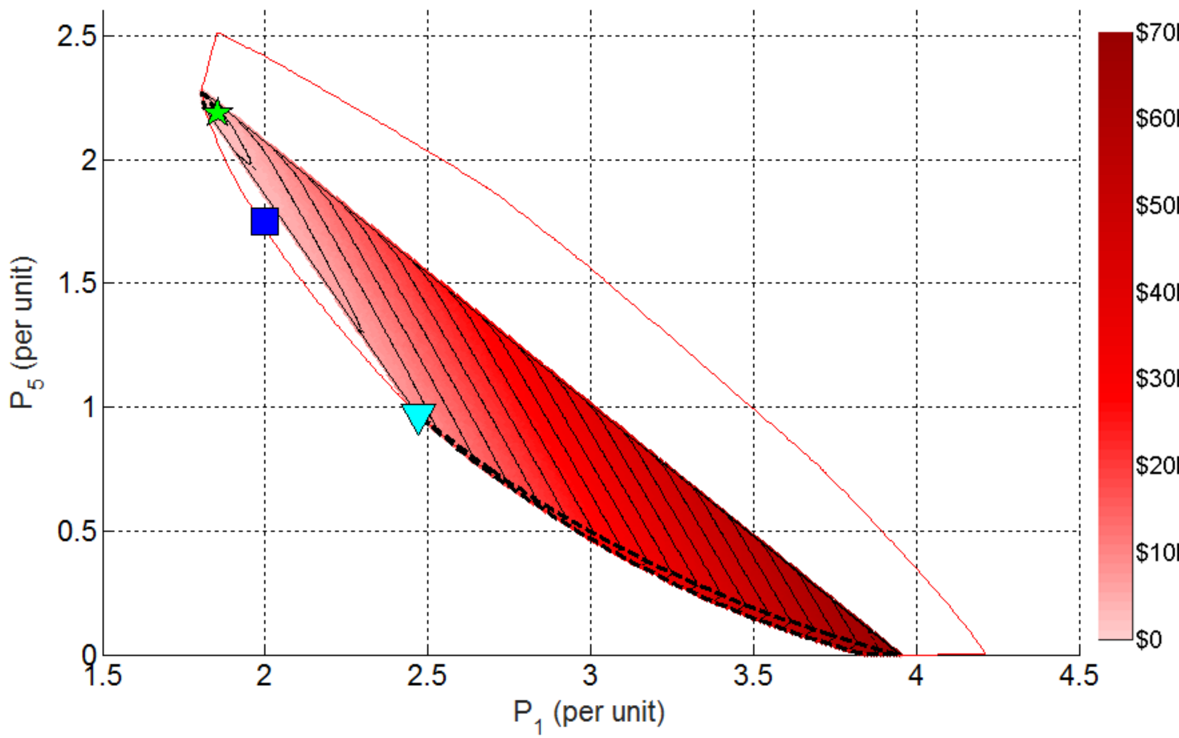


Feasible Space of Second-Order Moment Relaxation



Generalization of the SDP relaxation finds the global solution

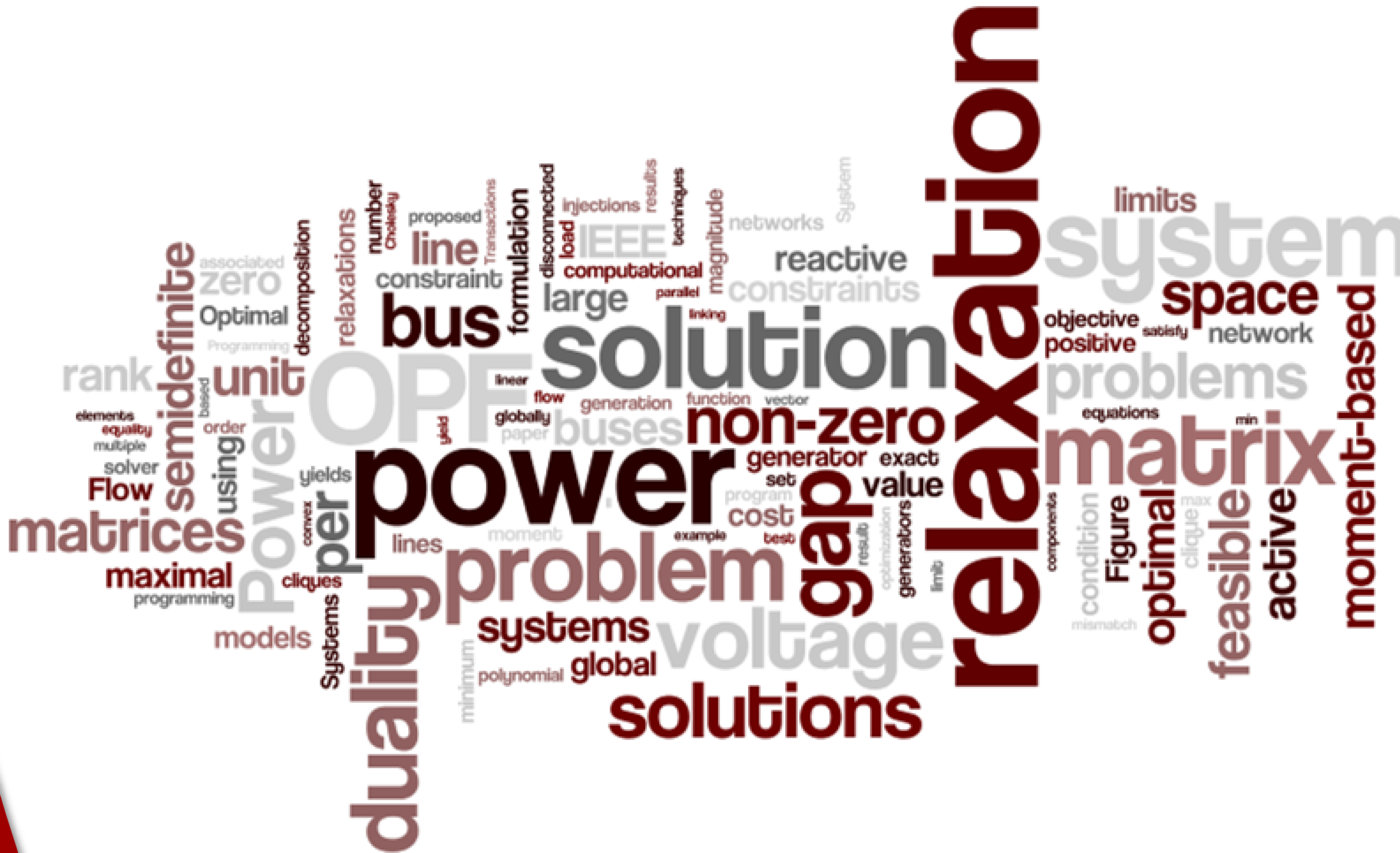
Feasible Space Algorithm

Conclusion

Conclusion

- The difficulty of solving OPF problems depends on the geometry of the associated feasible spaces
- Proposed a new approach for computing OPF feasible spaces
- Future work: compute feasible spaces for modified OPF formulations
  – Study difficulty imposed by various aspects of OPF problems, e.g., generator capability curves, line additions/outages, etc.

Questions?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