An Affine Arithmetic Method to Solve Stochastic Optimal Power Flow Problems

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Motivation

- Increased focus on renewable generation and many concerns in planning and operation of modern power systems
- Margins of operations for thermal generators to provide system reliability and efficiency
- A wide variety of probabilistic methods to incorporate uncertainties due to renewable sources integration
  - Monte-Carlo Simulation (MCS)
  - Analytical Methods (Convolution methods)
  - Probabilistic Methods (PDF)
  - Self Validated Computation Methods (SVC)
    - Interval Arithmetic (IA)
    - Affine Arithmetic (AA)
Research Objectives

• Develop an accurate and efficient AA-based OPF model to incorporate uncertainties in power systems.
  – Validate the AA-based operation system models with the MCS based method.

• Use the resulting AA based intervals to estimate the spinning reserve requirements in the presence of high DG penetration without PDFs.

• Test and validate the model on small and large systems.
Self-Validated Computation Method

• Most of the uncertainty analysis techniques, such as the MCS method, only capture external uncertainties.

• SVC methods keep track of internal errors inherently.
SVC Methods: Interval Arithmetic (IA)

- Considers internal and external errors, and provides the most conservative bounds

\[
\hat{x} + \hat{y} = [\underline{x} + \underline{y}, \bar{x} + \bar{y}]
\]
\[
\hat{x} - \hat{y} = [\underline{x} - \bar{y}, \bar{x} - \underline{y}]
\]
\[
\hat{x} \cdot \hat{y} = \left[ \min \{\underline{x} \cdot \underline{y}, \underline{x} \cdot \bar{y}, \bar{x} \cdot \underline{y}, \bar{x} \cdot \bar{y}\}, \max \{\underline{x} \cdot \underline{y}, \underline{x} \cdot \bar{y}, \bar{x} \cdot \underline{y}, \bar{x} \cdot \bar{y}\} \right]
\]
\[
\frac{\hat{x}}{\hat{y}} = [\underline{x}, \bar{x}].[1/\bar{y}, 1/\underline{y}]
\]

- Disadvantages of IA:
  Dependency problem
  Overflow problem
  Error explosion
SVC Methods: Affine Arithmetic (AA)

• It is an enhanced model for self validated numerical modeling, in which the quantities of interests presented as affine forms of certain primitive variables.

• It keeps track of correlations between computed and input quantities

• Affine representation of a value:

\[ \tilde{x} = x_0 + x_1 \varepsilon_1 + x_2 \varepsilon_2 + \cdots + x_n \varepsilon_n \]

• Interval range:

\[ [\tilde{x}] = \left[ x_0 - \sum_i |x_i| , x_0 + \sum_i |x_i| \right] \]
AA vs. IA

\[
\tilde{x} = 10 + 2\varepsilon_1 + \varepsilon_2 - \varepsilon_4 \\
\tilde{y} = 20 - 3\varepsilon_1 + \varepsilon_3 + \varepsilon_4
\]

\[
\hat{x} = [6, 14] \\
\hat{y} = [12, 28]
\]
Basic Affine Operations

\[ z = f(x, y) \rightarrow \tilde{z} = \tilde{f}(\tilde{x}, \tilde{y}) \]

\[ \tilde{x} \pm \tilde{y} = (x_0 \pm y_0) + (x_1 \pm y_1)\varepsilon_1 + \cdots + (x_n \pm y_n)\varepsilon_n \]

\[ \alpha \tilde{x} = (\alpha x_0) + (\alpha x_1)\varepsilon_1 + (\alpha x_2)\varepsilon_2 + \cdots + (\alpha x_n)\varepsilon_n \]

\[ \tilde{x} \pm \varphi = (x_0 \pm \varphi) + x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_n\varepsilon_n \]
Non-Affine Operations

\[ z = f(x, y) \rightarrow \tilde{z} = f(\tilde{x}, \tilde{y}) \]

\[
f^*(\varepsilon_1, \ldots, \varepsilon_n) = x_0 y_0 + \sum_{i=1}^{n} (x_0 y_i + y_0 x_i) \varepsilon_i + \sum_{i=1}^{n} x_i \varepsilon_i \sum_{i=1}^{n} y_i \varepsilon_i
\]

\[
\tilde{x}\tilde{y} = x_0 y_0 + \sum_{i=1}^{n} (x_0 y_i + y_0 x_i) \varepsilon_i + z_k \varepsilon_k
\]

\[
z_k = \sum_{i=1}^{n} |x_i| \sum_{i=1}^{n} |y_i|
\]
AA-based OPF model

\[
\begin{align*}
\min & \quad F(\bar{P}^G) = \sum_{i \in \text{Gas}} \alpha_i \bar{P}_i^G + \beta_i \bar{P}_i^G + c_i \\
\text{s.t.:} & \quad \Delta \bar{P}_i(\bar{e}_i, \bar{f}_i, \bar{I}_r_i, \bar{I}_{im_i}, \bar{P}_i^G, \bar{P}_i^D) = 0 \quad \forall i \in N \\
& \quad \Delta \bar{Q}_i(\bar{e}_i, \bar{f}_i, \bar{I}_r_i, \bar{I}_{im_i}, \bar{Q}_i^G, \bar{Q}_i^D) = 0 \quad \forall i \in N \\
& \quad |\bar{V}_i|^2 = \bar{e}_i^2 + \bar{f}_i^2 \quad \forall i \in N \\
& \quad P_{i_{\text{min}}} \leq \bar{P}_i^G \leq P_{i_{\text{max}}} \quad \forall i \in NPG \\
& \quad Q_{i_{\text{min}}} \leq \bar{Q}_i^G \leq Q_{i_{\text{max}}} \quad \forall i \in NPG \\
& \quad I_{ij_{\text{min}}} \leq \bar{I}_{ij} \leq I_{ij_{\text{max}}} \quad \forall ij \in L \\
& \quad V_{i_{\text{min}}} \leq |\bar{V}_i| \leq V_{i_{\text{max}}} \quad \forall i \in N
\end{align*}
\]
AA-based OPF model

\[ P_i^D = \frac{\bar{P}_i^D - p_i^D}{2}, \quad Q_i^D = \frac{\bar{Q}_i^D - q_i^D}{2} \]

Solve OPF

\[ e_{i0}, f_{i0} \]

Construct \( \bar{e}_i, \bar{f}_i \)

Perturb demand at bus \( i \in N \)

Find deviations from center values

\[ e_{i,j}^p, f_{i,j}^p \]
Bus Voltage Components in AA Form

\[
\tilde{e}_i = e_{i0} + \sum_{j \in N} e_{i,j}^P \varepsilon_{Pj} + \sum_{j \in N} e_{i,j}^Q \varepsilon_{Qj} \quad \forall i \in N
\]

\[
\tilde{f}_i = f_{i0} + \sum_{j \in N} f_{i,j}^P \varepsilon_{Pj} + \sum_{j \in N} f_{i,j}^Q \varepsilon_{Qj} \quad \forall i \in N
\]

\[
e_{i,j}^P = \left. \frac{\partial e_i}{\partial P_j} \right|_0 \approx \frac{e_i^N - e_i^0}{\Delta P_j} \quad \forall i, j \in N
\]

\[
e_{i,j}^Q = \left. \frac{\partial e_i}{\partial Q_j} \right|_0 \approx \frac{e_i^N - e_i^0}{\Delta Q_j} \quad \forall i, j \in N
\]

\[
f_{i,j}^P = \left. \frac{\partial f_i}{\partial P_j} \right|_0 \approx \frac{f_i^N - f_i^0}{\Delta P_j} \quad \forall i, j \in N
\]

\[
f_{i,j}^Q = \left. \frac{\partial f_i}{\partial Q_j} \right|_0 \approx \frac{f_i^N - f_i^0}{\Delta Q_j} \quad \forall i, j \in N
\]
Affine Real and Reactive Power

\[ \tilde{e}_i, \tilde{f}_i \]

Affine Operations

Calculate \( \tilde{I}_{r_i}, \tilde{I}_{im_i} \)

Calculate \( \tilde{P}_i, \tilde{Q}_i \)
Reactive power $\tilde{P}$ and $\tilde{Q}$ are calculated as follows:

$$\tilde{P} = \bar{e} \tilde{I}_r + \bar{f} \tilde{I}_{im}$$

$$\tilde{Q} = \bar{f} \tilde{I}_r - \bar{e} \tilde{I}_{im}$$

$\tilde{P}_i$ and $\tilde{Q}_i$ have the following affine forms:

$$\tilde{P}_i = P_{i,0} + \sum_j P_{i,j}^\epsilon \epsilon_{P_j} + \sum_j P_{i,j}^Q \epsilon_{Q_j} + P_i^T \epsilon_T$$

$$\tilde{Q}_i = Q_{i,0} + \sum_j Q_{i,j}^\epsilon \epsilon_{P_j} + \sum_j Q_{i,j}^Q \epsilon_{Q_j} + Q_i^T \epsilon_T$$
Contraction Method

\[
\begin{align*}
\text{min} & \quad \tilde{F} \left( \varepsilon_{P_j}, \varepsilon_{Q_j} \right) \\
\text{s.t.:} & \quad \Delta \tilde{P}_i \left( \varepsilon_{P_j}, \varepsilon_{Q_j} \right) = 0 & \forall i \in N \\
& \quad \Delta \tilde{Q}_i \left( \varepsilon_{P_j}, \varepsilon_{Q_j} \right) = 0 & \forall i \in N \\
& \quad p_i^{\text{min}} \leq \tilde{P}_i \left( \varepsilon_{P_j}, \varepsilon_{Q_j} \right) \leq p_i^{\text{max}} & \forall i \in NPG \\
& \quad q_i^{\text{min}} \leq \tilde{Q}_i \left( \varepsilon_{P_j}, \varepsilon_{Q_j} \right) \leq q_i^{\text{max}} & \forall i \in NPG \\
& \quad I_{ij}^{\min^2} \leq I_{rij}^2 \left( \varepsilon_{P_j}, \varepsilon_{Q_j} \right) + \tilde{I}_{imij}^2 \left( \varepsilon_{P_j}, \varepsilon_{Q_j} \right) \leq I_{ij}^{\text{max}^2} & \forall ij \in L \\
& \quad V_i^{\min^2} \leq \tilde{V}_i^2 \left( \varepsilon_{P_j}, \varepsilon_{Q_j} \right) \leq V_i^{\text{max}^2} & \forall i \in N
\end{align*}
\]
Contraction Method

\[ \tilde{P}_i, \tilde{Q}_i, \tilde{I}_r_i, \tilde{I}_m_i, \tilde{e}_i, \tilde{f}_i \]

Contraction Method

Solve LP problem

\[ \varepsilon_{P_i}^D, \varepsilon_{Q_i}^D \]

Calculate P and Q intervals
Contraction Method

- AA is implemented in GAMS, for IEEE 30-bus benchmark system and a real 1211-bus European system.
- Results are compared with Monte-Carlo simulation.
- MCS uses:
  - 3000 iterations
  - Uniform distribution
AA v.s. MCS Real and Reactive Power Ranges

![Graph showing real and reactive power generation](image-url)

**Real power generation (p.u.)**

- PGupMC
- PGloMC
- PGupAA
- PGloAA

**Reactive power generation (p.u.)**

- QGupMC
- QGloMC
- QGupAA
- QGloAA

Bus number

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FERC Conference
AA v.s. MCS Bus Voltage Magnitude Intervals

Bus Voltage Magnitude (p.u.)

Bus number

VupMC, VloMC, VupAA, VloAA
AA v.s. MCS Real and Reactive Power (Thermals)

% of deviation from center value

Buses

0 10 20 30 40 50 60 70

% of deviation (PG-AA)

% of deviation (PG-MC)

% of deviation from center value

Buses

0 10 20 30 40 50 60 70 80 90

% of deviation (QG-AA)

% of deviation (QG-MC)
AA v.s. MCS Bus Voltage Magnitudes

% of deviation from center value

buses

V-AA
V-MC
### Results: Thermal Generators’ output range

<table>
<thead>
<tr>
<th>Total Thermal Reserve</th>
<th>AA-based Method (GW)</th>
<th>MCS-based Method (GW)</th>
<th>% of Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum</td>
<td>143.9</td>
<td>142.6</td>
<td>0.91%</td>
</tr>
<tr>
<td>Minimum</td>
<td>131.1</td>
<td>128.2</td>
<td>2.21%</td>
</tr>
</tbody>
</table>
Future Work

- Develop a computationally efficient and numerically accurate AA-based model to solve the stochastic UC model with intermittent sources of energy such as wind and solar.

- Provide a detailed comparison of the AA-based solution with the other available solutions such as MCS in regards to its computational efficiency and numerical accuracy.

- Use AA-based method to develop local marginal price intervals, due to uncertainties in the system.
Thanks

Jacobian-Based Sensitivity Analysis

\[ f(x, p) = 0 \]

\[ \frac{\partial f(x_0)}{\partial x} dx + \frac{\partial f(x_0)}{\partial p} dp = 0 \]

\[ \frac{\Delta x}{\Delta p} \approx \frac{dx}{dp} = - \left[ \frac{\partial f(x_0)}{\partial x} \right]^{-1} \frac{\partial f(x_0)}{\partial p} \]
- An approximation that minimize the maximum absolute error.

- If $f$ is a bounded and continuous function from some closed and bounded interval $I = [a, b]$, then the Chebyshev approximation is $\alpha x + \zeta$

  - $\alpha = \frac{f(b) - f(a)}{b - a} = f'(u)$
  - $\zeta$ is such that $\alpha u + \zeta = (f(u) + r(u))/2$
**AA-Based PF Formula**

### Affine Current Calculations:

\[
\tilde{I} = Y |\tilde{V}|
\]

\[
\tilde{I} = (G + jB)(\tilde{e} + j\tilde{f})
\]

\[
\tilde{I}_r = G\tilde{e} - B\tilde{f}
\]

\[
\tilde{I}_{im} = G\tilde{f} + B\tilde{e}
\]

\[
\tilde{I}_{ri} = I_{ri,0} + \sum_{j \in N} I_{ri,j}^p \varepsilon_{p,j} + \sum_{j \in N} I_{ri,j}^q \varepsilon_{q,j} \quad \forall i, j \in N
\]

\[
\tilde{I}_{im} = I_{im,0} + \sum_{j \in N} I_{im,i,j}^p \varepsilon_{p,j} + \sum_{j \in N} I_{im,i,j}^q \varepsilon_{q,j} \quad \forall i, j \in N
\]

### Affine Real and Reactive Power calculation:

\[
P_{i,0} = e_{i,0} I_{ri,0} + f_{i,0} I_{im,0}
\]

\[
Q_{i,0} = f_{i,0} I_{ri,0} - e_{i,0} I_{im,0}
\]

\[
P_{i,j}^p = e_{i,0} I_{ri,j}^p + I_{ri,0} e_{i,j}^p + f_{i,0} I_{im,j}^p + I_{im,0} f_{i,j}^p
\]

\[
Q_{i,j}^p = f_{i,0} I_{ri,j}^p + I_{ri,0} f_{i,j}^p - e_{i,0} I_{im,j}^p - I_{im,0} e_{i,j}^p
\]

\[
P_i^T = \sum_j |e_{i,j}^p| \sum_j |I_{ri,j}^p| + \sum_j |f_{i,j}^p| \sum_j |I_{im,j}^p|
\]

\[
Q_i^T = \sum_j |f_{i,j}^p| \sum_j |I_{ri,j}^p| - \sum_j |e_{i,j}^p| \sum_j |I_{im,j}^p|
\]

\[
\bar{P}_i = P_{i,0} + \sum_j P_{i,j}^p \varepsilon_{p,j} + \sum_j P_{i,j}^q \varepsilon_{q,j} + P_i^T \varepsilon_{T_i}
\]

\[
\bar{Q}_i = Q_{i,0} + \sum_j Q_{i,j}^p \varepsilon_{q,j} + \sum_j Q_{i,j}^q \varepsilon_{q,j} + Q_i^T \varepsilon_{T_i}
\]
Probabilistic power injection is a function of random variables, generation and load thus the result is a random variable with a PDF

### Univariate function

\[ y = g(x) \]

\[ P(Y \leq y) = P[X \leq g(y)^{-1}] \]

\[ F_Y(y) = F_X(g(y)^{-1}) \]

\[ = \int_{-\infty}^{g(y)^{-1}} f_X(x) \, dx \]

\[ f_Y(y) = f_X(g(y)^{-1}) \frac{dg^{-1}}{dy} \]

**Example:** \( Z = x + y \)

\[ f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z - y, y) \, dy \]

\[ f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y) \, dy \]

### Bivariate function

\[ z = g(x, y) \]

\[ x = g^{-1}(z, y) \text{ and } y = g^{-1}(x, z) \]

\[ F_Z(z) = P(g(x, y) \leq z) = P[X \leq g(y)^{-1}] \]

\[ F_Z(z) = F_Z(g(x, y) \leq z) \]

\[ = \int_{-\infty}^{g^{-1}(z, y)} f_X(x, y) \, dx \, dy = \int_{-\infty}^{z} \int_{-\infty}^{g^{-1}(y)} f_{XY}(x, y) \left| \frac{dg^{-1}}{dy} \right| \, dy \, dz \]

\[ f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(g^{-1}, y) \left| \frac{dg^{-1}}{dy} \right| \, dy \]

\[ f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y) \, dy \]