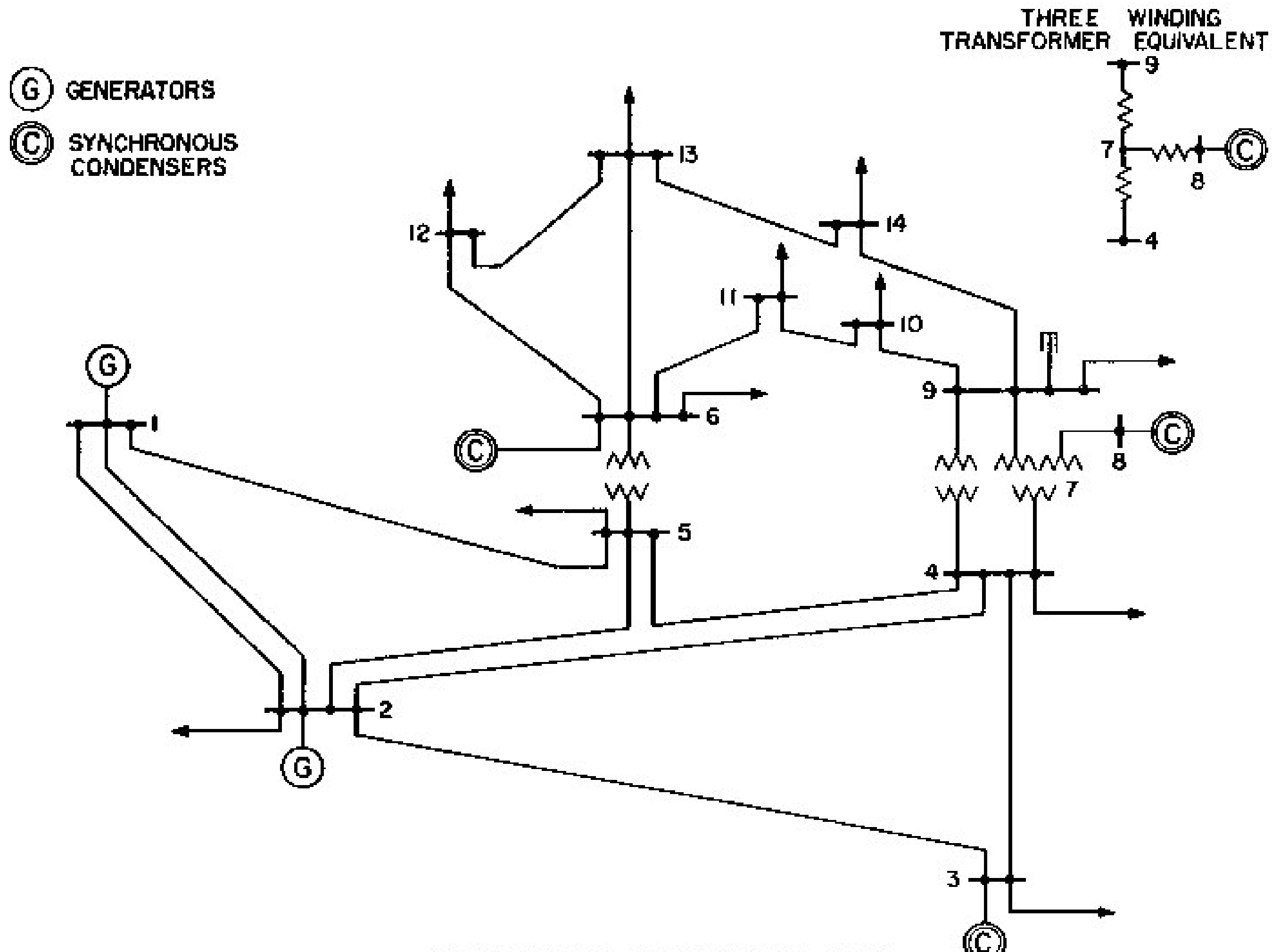
Conic Relaxations of Alternating Current Optimal Power Flow (ACOPF)

Alper Atamtürk, Chen Chen, Shmuel Oren,
University of California, Berkeley
Dr. Richard P. O'Neill, *FERC*

FERC Software Conference

June 26, 2012

Single-line diagram



AEP 14 BUS TEST SYSTEM BUS CODE DIAGRAM

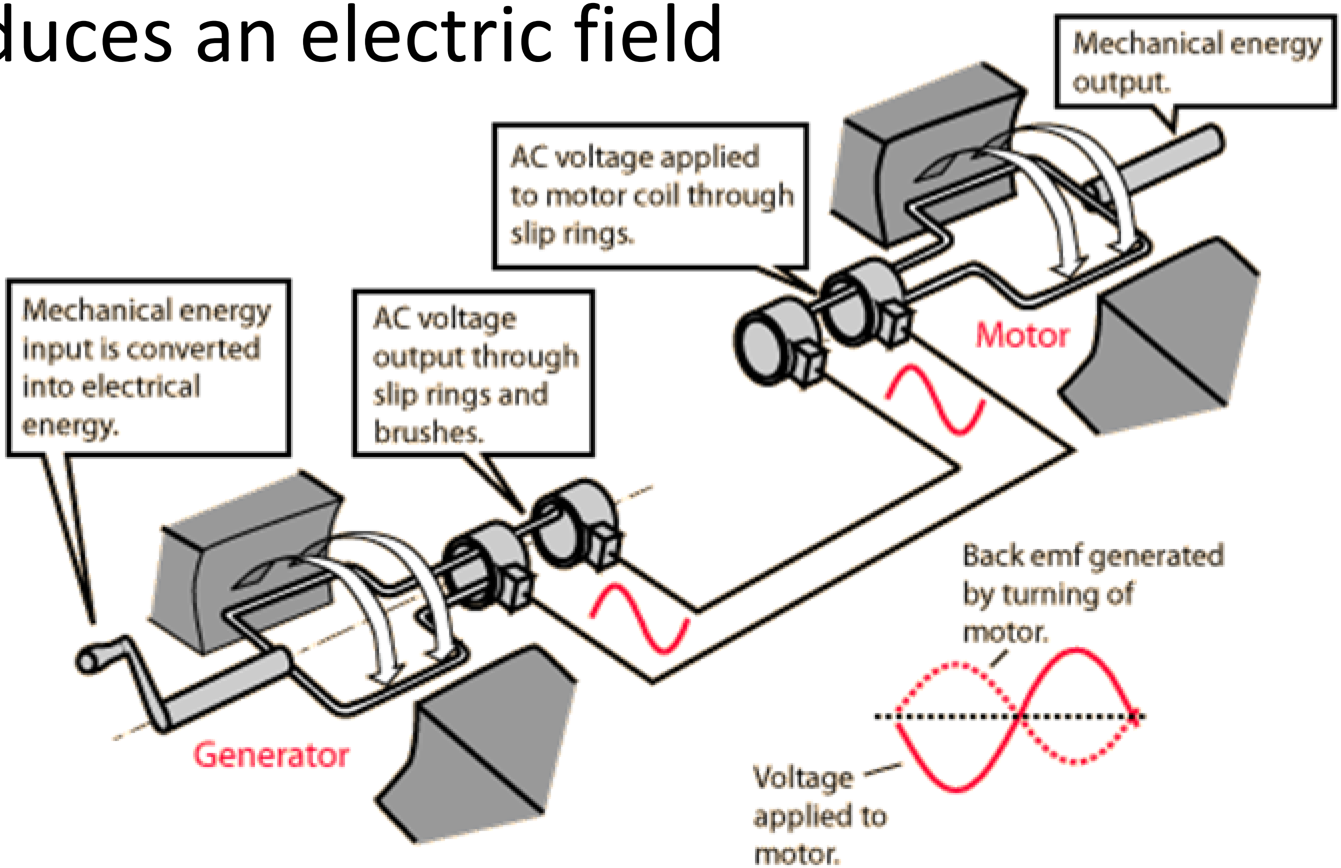
Alternating Current Optimal Power Flow

- Variables: generators, some network elements
- Objective: e.g. minimize costs
- Constraints: balance supply & demand, network constraints, generator constraints

- Obstacle: power flow is dynamic, nonlinear

Complex Flow

- Faraday's law: a changing magnetic field induces an electric field



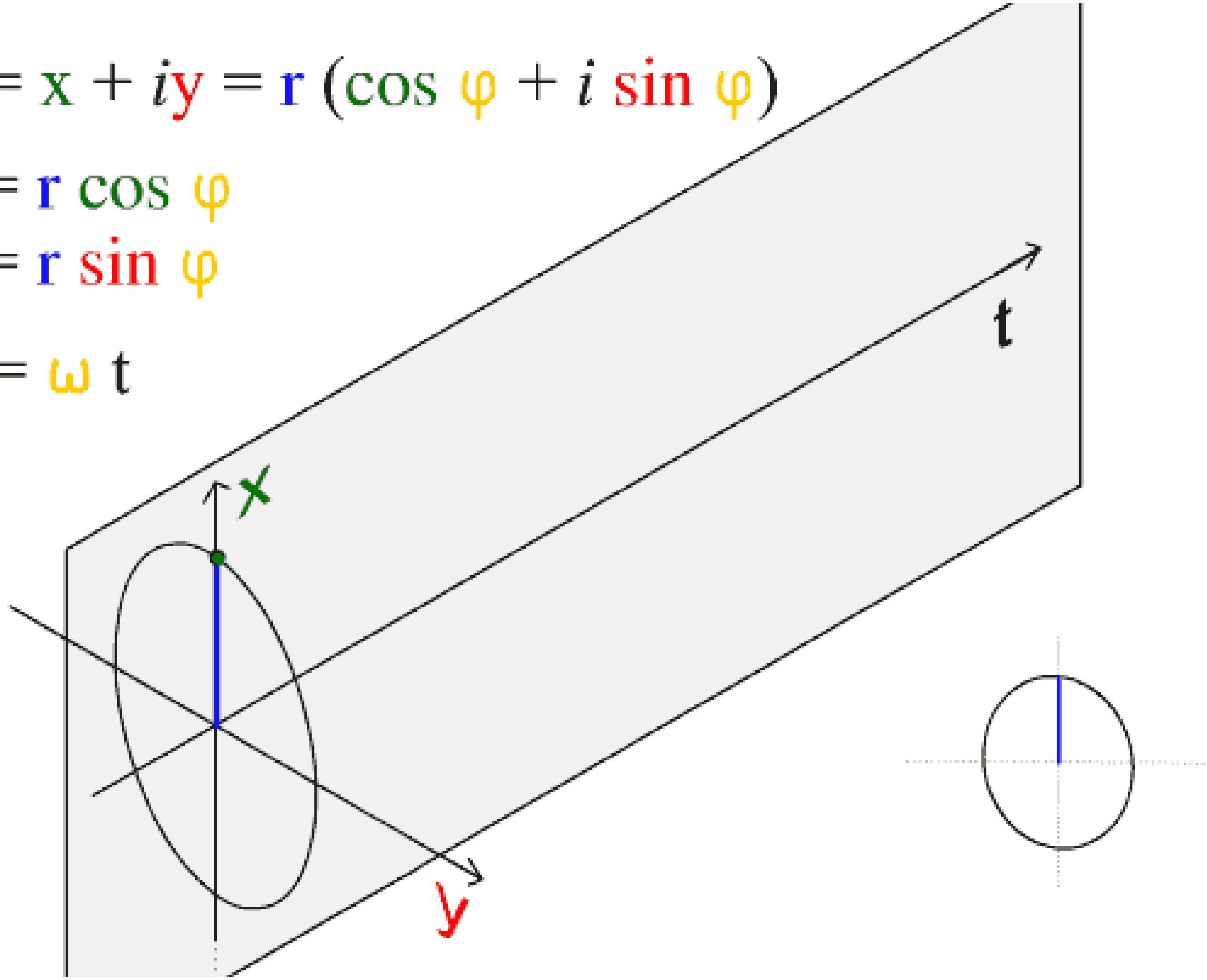
Alternating Current (AC)

$$z = x + iy = r\,(\cos\varphi + i\sin\varphi)$$

$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$\varphi = \omega t$$



Voltage Phasors

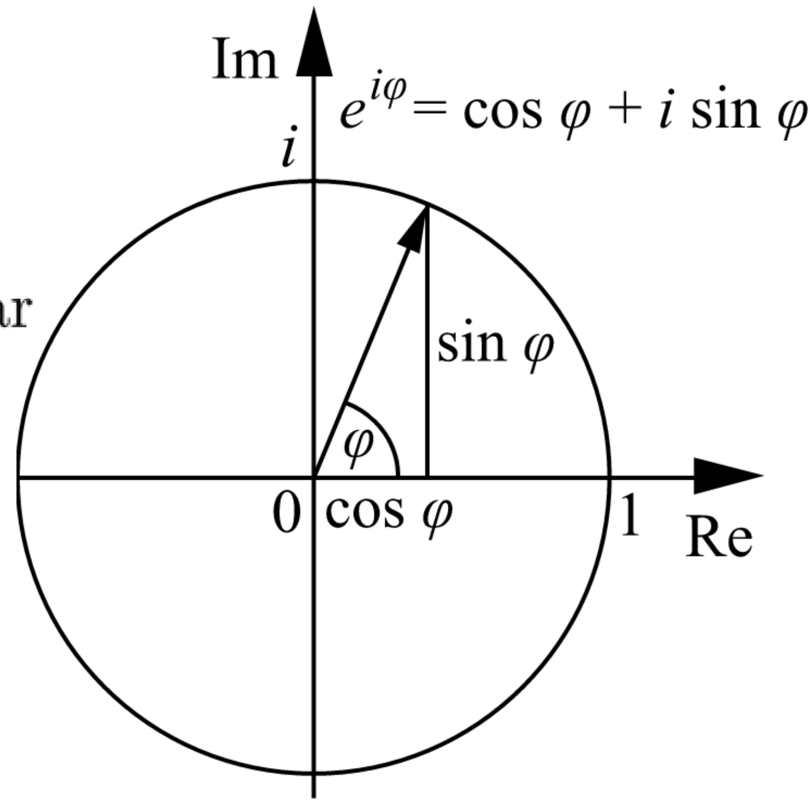
Wave: ~~Frequency~~, Amplitude, Phase

$$|V| \triangleq \frac{V_{\max}}{\sqrt{2}}$$

$$V = |V|e^{j\theta} \text{ Exponential}$$

$$= |V|\angle\theta \text{ Polar}$$

$$= |V|\cos(\theta) + j|V|\sin(\theta) \text{ Rectangular}$$



ACOPF: Polar Formulation

$$\min \sum_{i} c_1 P_i^2 + c_2 P_i$$

subject to

$$P_i = \sum_{\forall n} [G_{in}|V_i||V_n| \cos(\theta_i - \theta_n) + |V_i||V_n| B_{in} \sin(\theta_i - \theta_n)], \forall i \quad NPV(1)$$

$$Q_i = \sum_{\forall n} [G_{in}|V_i||V_n| \sin(\theta_i - \theta_n) - |V_i||V_n| B_{in} \cos(\theta_i - \theta_n)], \forall i \quad NPV(2)$$

$$P^{\min} \leq P \leq P^{\max} \quad NPV(3)$$

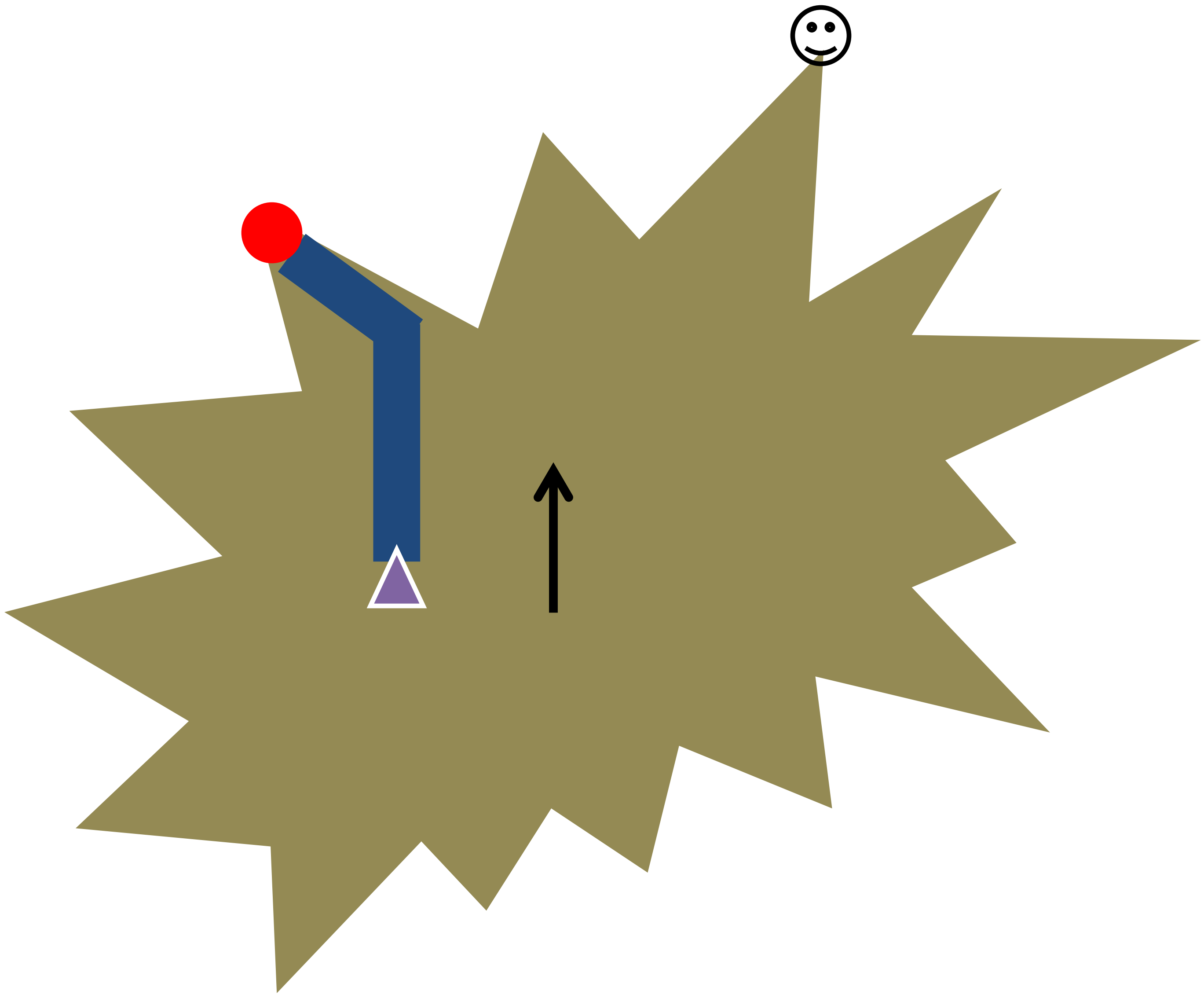
$$Q^{\min} \leq Q \leq Q^{\max} \quad NPV(4)$$

$$V^{\min} \leq |V| \leq V^{\max} \quad NPV(5)$$

$$-\theta_{in}^{\max} \leq \theta_i - \theta_n \leq \theta_{in}^{\max}, \forall i, n \in B \quad NPV(6)$$

$$\sqrt{(P_i - P_n)^2 + (Q_i - Q_n)^2} \leq S_{in}^{\max}, \forall i, n \in B \quad NPV(7)$$

The Trouble with Nonconvexity



Linear Approximations

Can easily find an exact solution to an approximate, but inexact ACOPF

PTDF: linear network flow problem derived from "typical" conditions

B-Theta aka DCOPF:

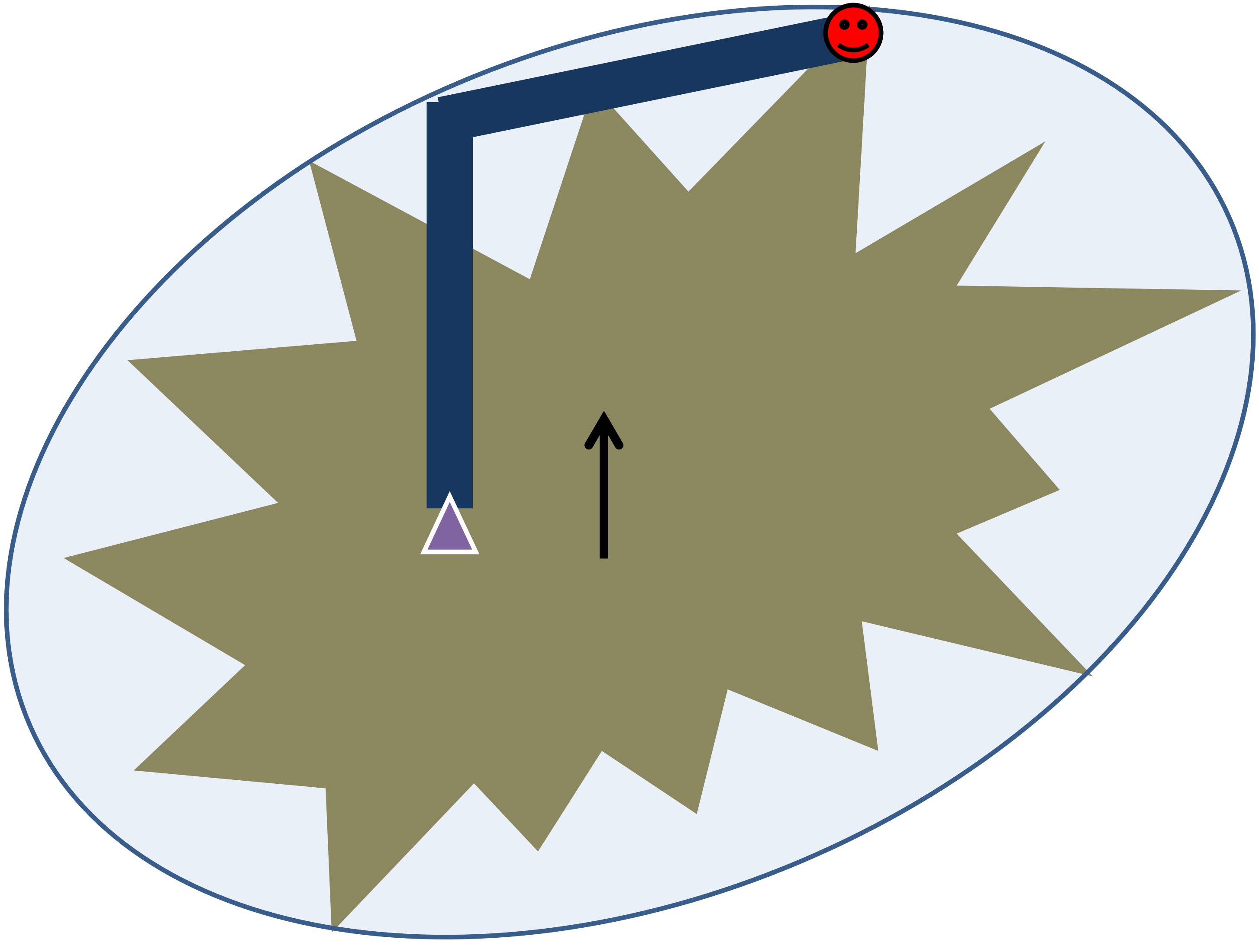
- Assume $|V| = 1$, $\Delta\theta \sim 0$, $G << B$
- Disregard reactive power: $Q \sim 0$

$$P_i \approx B_{in}(\theta_i - \theta_n)$$

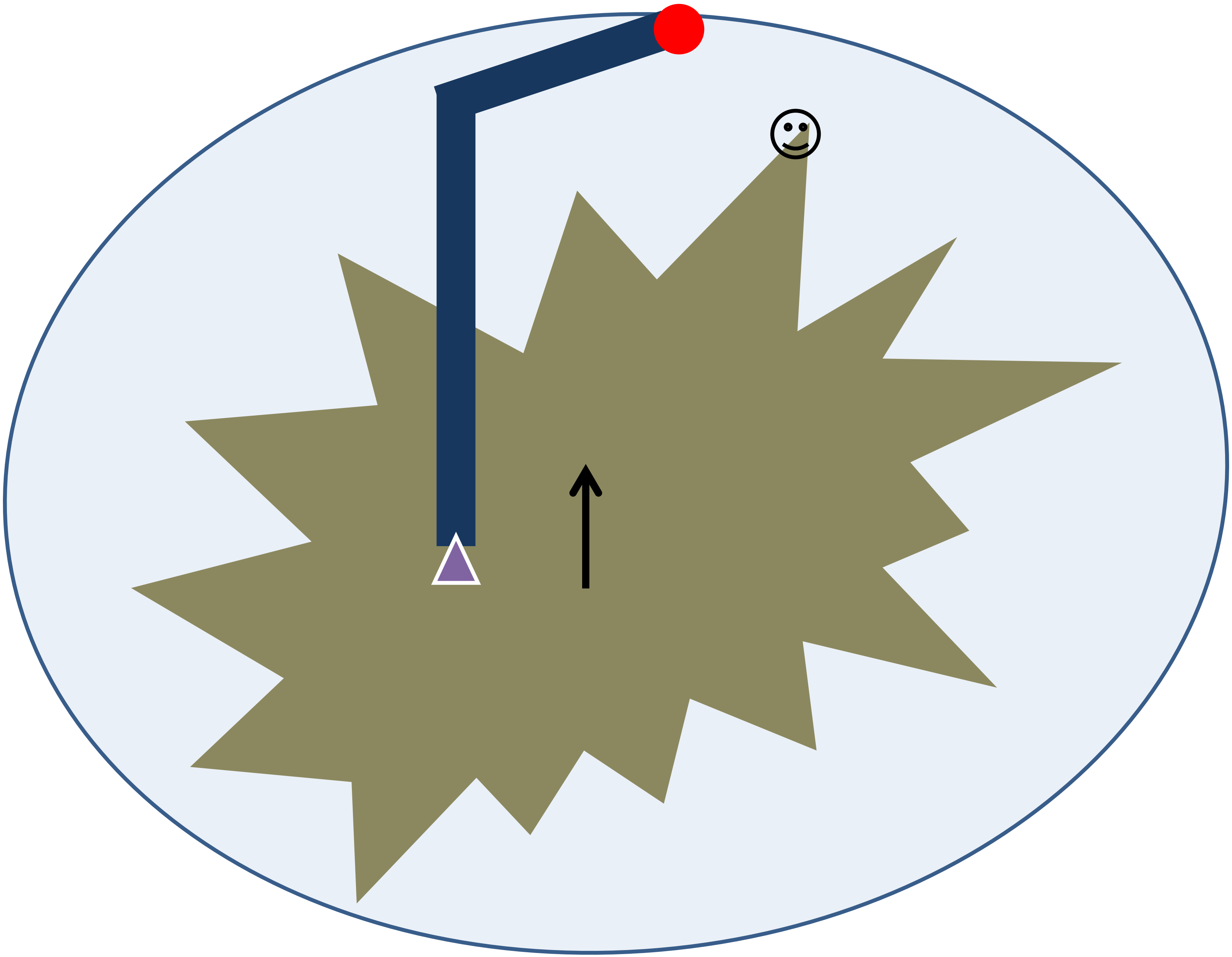
Conic Relaxations

- Relaxation Feasibility Region $\supseteq$ Original Region
  - Relaxation helps determine optimality/infeasibility
- Conic programs use special convex constraints:
  - LP: Linear (nonnegativity)
  - SOCP: Second Order Cones (convex quadratics)
  - SDP: Semidefinite (positive semidefiniteness)
  - SDP $\supset$ SOCP $\supset$ LP
- Conic programs are relatively easy
  - Near-optimal solutions in polynomial time
  - Starting points/infeasibility detection

Convex Relaxation



Inexact Relaxation



ACOPF: Rectangular Voltage

Nodes i={1,...,N}, voltages $V = \{V_1^{\text{Re}}, ..., V_N^{\text{Re}}, V_1^{\text{Im}}, ..., V_N^{\text{Im}}\}$

Recall $V^{\text{Re}} = |V|\cos(\theta), V^{\text{Im}} = |V|\sin(\theta) \Rightarrow \frac{V^{\text{Im}}}{V^{\text{Re}}} = \tan(\theta)$

$$\min \sum_i c_1 P_i^2 + c_2 P_i$$

$s.t.$

$$P_i = V^T Y_{P_i} V \qquad \forall i \quad (1)$$

$$P_i^{\min} \leq P_i \leq P_i^{\max} \qquad \forall i \quad (2)$$

$$Q_i = V^T Y_{Q_i} V \qquad \forall i \quad (3)$$

$$Q_i^{\min} \leq Q_i \leq Q_i^{\max} \qquad \forall i \quad (4)$$

$$V_{\min} \leq (V_i^{\text{Re}})^2 + (V_i^{\text{Im}})^2 \leq V_{\max} \qquad \forall i \quad (5)$$

$$(P_i - P_n)^2 + (Q_i - Q_n)^2 \leq s_{\text{in}}^2 \qquad \forall i, n \quad (6)$$

$$\theta_{\text{in}}^{\min} \leq \arctan\left(\frac{V_i^{\text{Im}}}{V_i^{\text{Re}}}\right) - \arctan\left(\frac{V_n^{\text{Im}}}{V_n^{\text{Re}}}\right) \leq \theta_{\text{in}}^{\max} \qquad \forall i, n \quad (7)$$

NVR_SDP
(Primal SDP Relaxation of NVR formulation)

$$\min \sum_i c_1 P_i^2 + c_2 P_i$$

subject to

$$P_i = \sum_{n \in N} [G_{in}(X_{V_i^{Re} V_n^{Re}}, + X_{V_i^{Im} V_n^{Im}}) - B_{in}(X_{V_i^{Re} V_n^{Im}} - X_{V_i^{Im} V_n^{Re}})] \; \forall i$$

$$Q_i = \sum_{n \in N} [-B_{in}(X_{V_i^{Re} V_n^{Re}}, + X_{V_i^{Im} V_n^{Im}}) - G_{in}(X_{V_i^{Re} V_n^{Im}} - X_{V_i^{Im} V_n^{Re}})] \; \forall i$$

$$P^{\min} \leq P \leq P^{\max} \quad NRV\_SDP(3)$$

$$Q^{\min} \leq Q \leq Q^{\max} \quad NRV\_SDP(4)$$

$$V^{\min} \leq |V| \leq V^{\max} \quad NRV\_SDP(5)$$

$$\sqrt{(P_i - P_n)^2 + (Q_i - Q_n)^2} \leq S_{in}^{\max}, \forall i, n \in B \quad NRV\_SDP(6)$$

$$X \succeq VV^T \equiv X \succeq 0 \quad NRV\_SDP(7)$$

NVR_SOCP
(SOCP Relaxation of NVR_SDP)

All 2x2 principal minors must be nonnegative

$$X \succeq 0 \implies X_{ij}^2 \leq X_{ii} X_{jj}$$

NBR_SOCP

$$\min \sum_{i} c_1 P_i^2 + c_2 P_i$$

subject to

$$P_i = \sqrt{2} G_{ii} u_i + \sum_{n \in N \setminus \{i\}} [G_{in} M_{in} + B_{in} T_{in}] \; \forall i \quad NBR\_SOCP(1)$$

$$Q_i = -\sqrt{2} B_{ii} u_i + \sum_{n \in N \setminus \{i\}} [G_{in} T_{in} + B_{in} M_{in}] \; \forall i \quad NBR\_SOCP(2)$$

$$P^{\min} \leq P \leq P^{\max} \quad NBR\_SOCP(3)$$

$$Q^{\min} \leq Q \leq Q^{\max} \quad NBR\_SOCP(4)$$

$$\frac{(V^{\min})^2}{\sqrt{2}} \leq u \leq \frac{(V^{\max})^2}{\sqrt{2}} \quad NBR\_SOCP(5)$$

$$\sqrt{(P_i - P_n)^2 + (Q_i - Q_n)^2} \leq S_{in}^{\max}, \forall i, n \in B \quad NBR\_SOCP(7)$$

$$2 u_i u_n \geq M_{in}^2 + T_{in}^2 \; \forall i, n \in B \quad NBR\_SOCP(8)$$

$$M, u \geq 0 \quad NBR\_SOCP(9)$$

Comparing SOCP Relaxations: Relaxation Gaps

(Zero Lagrangian dual/SDP relaxation gap)

System name	NRV_SOCP	NBR_SOCP
Case9	1.45%	0%
Case14	5.22%	0.08%
Case_ieee30	5.86%	0.04%
Case30	1.81%	0.58%
Case39	1.40%	0.02%
Case57	1.57%	0.08%
Case118	49.68%	39.53%
Case300	2.05%	0.34%

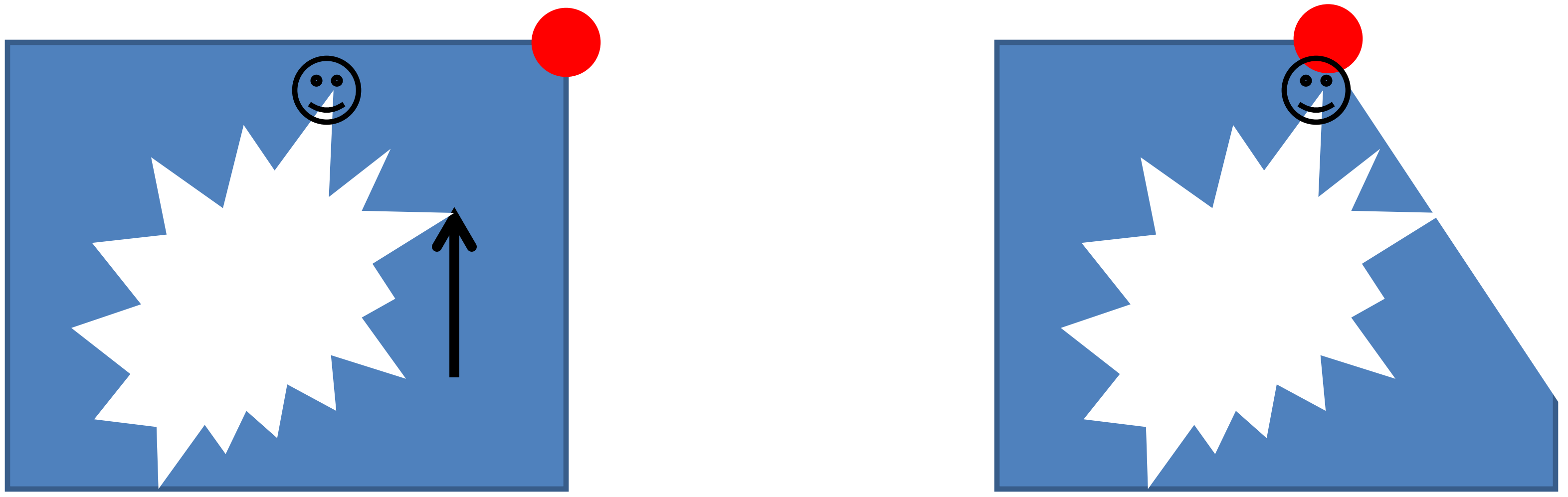
$$\text{Relaxation Gap} = 1 - \frac{\text{Relaxation Optimal Value}}{\text{Best Known Primal Objective}}$$

Comparing SOCP Relaxations: Solution Times (sec)

System name	NRV_DSDP	NRV_SDP	NRV_SOCP	NBR_SOCP
Case9	2.8	1.4	0.01	0.01
Case14	5.6	2.6	0.03	0.02
Case_ieee30	5.5	56.7	0.06	0.04
Case30	2.6	64.5	0.06	0.05
Case39	12.3	446	0.08	0.04
Case57	11.6	2478.6	0.13	0.04
Case118	49.36	-	0.83	0.1
Case300	749.9	-	1.5	0.3

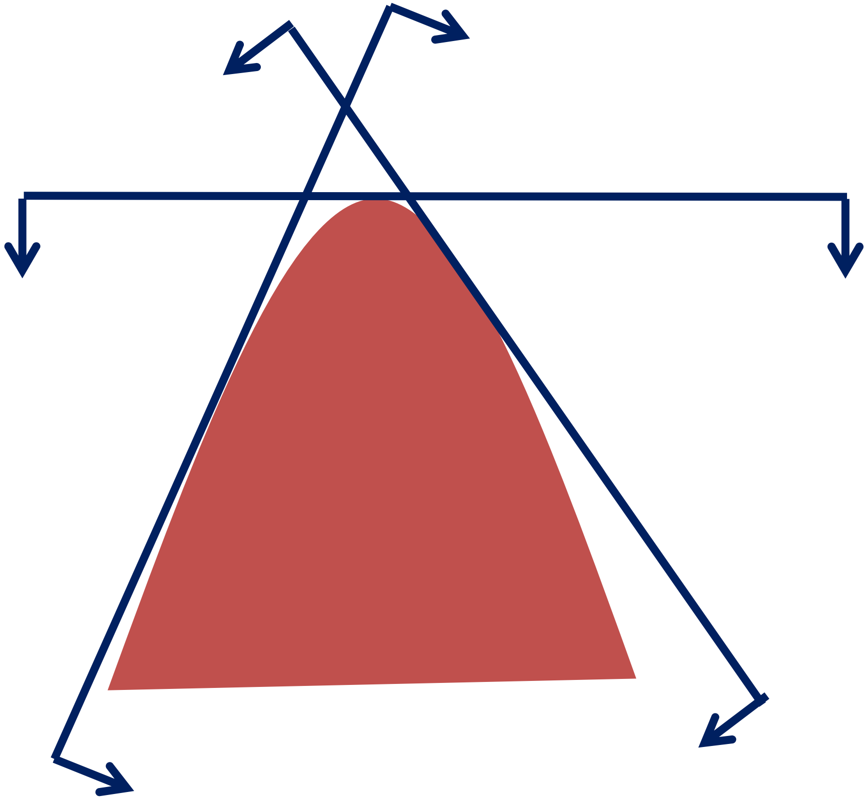
Cuts

A cut is a constraint that renders infeasible a (usually optimal) point of an inexact relaxation



Solving SDP as SOCP

$$X \succeq 0 \equiv c^T X c \geq 0 \quad \forall c \in \mathbb{R}$$



Cuts for NVR_SOCP: case118

iteration	# of cuts	gap	time to solve (sec)
0	-	49.3	0.5
1	80	52.5	4.8
2	80	55.7	7.2
3	78	99.6	11

Duality Gap

- Can cuts help us with problematic cases?
  - Stressed, possibly infeasible systems
  - High congestion? Negative costs?

Using Disjunctions for Nonconvexity

Divide nonconvex space into linear sections

Given polytope $P = \{x \in \mathbb{R}^n | Ax \geq b\}$

and disjunction $D = \vee_{k=1}^{q} (D_k x \geq d_k)$

For some point EITHER

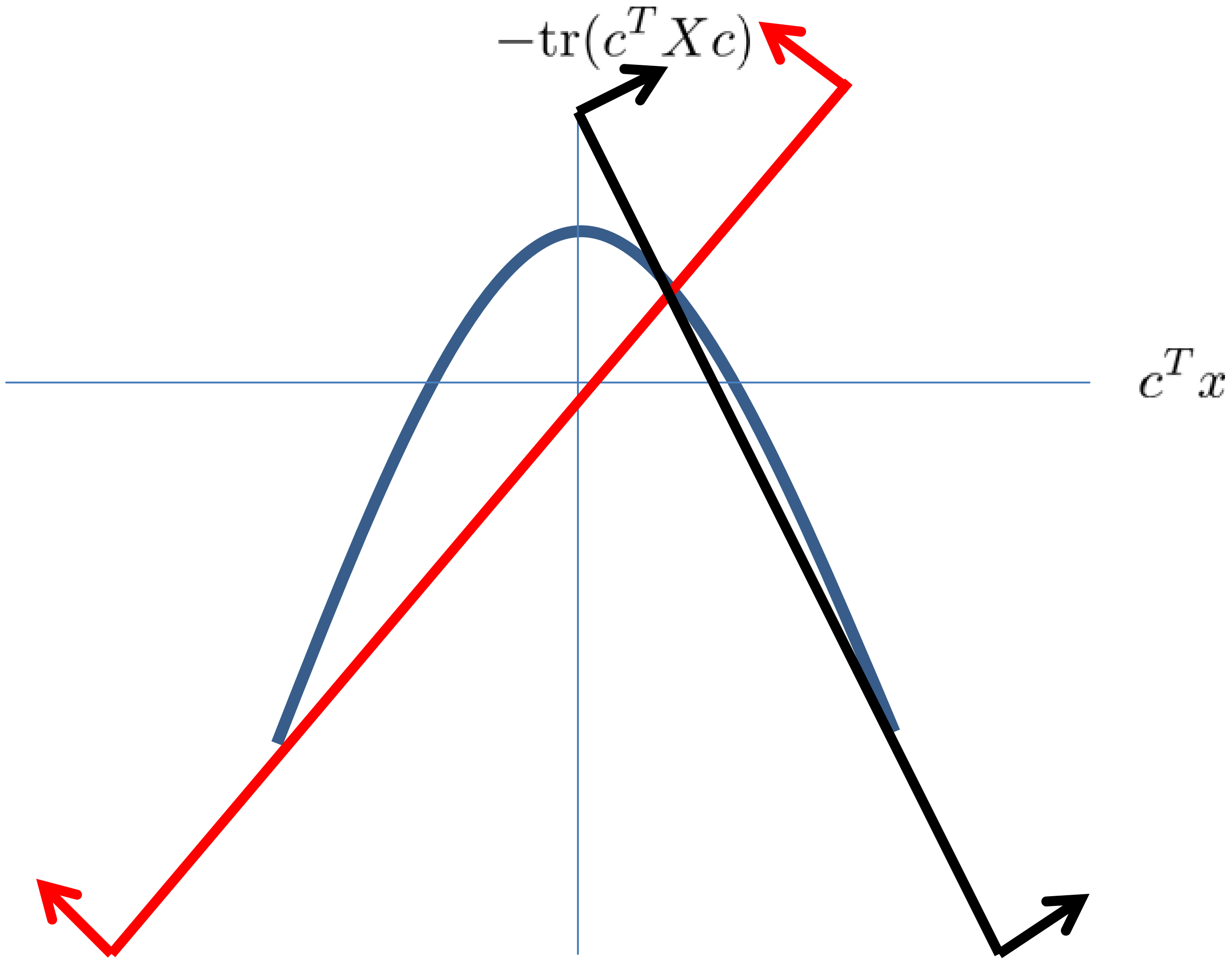
$\hat{x} \in Q = \text{clconv} \cup_{k=1}^{q} \{x \in P | D_k x \geq d_k\}$

OR separate: find a valid inequality/cut s.t.:

$\alpha^T x \geq \beta \mid \alpha^T \hat{x} < \beta$

Cut-Generating Conic Program handles mechanics

Nonconvex Relaxations with Disjunctions



Valid cuts for $X = xx^T \implies (c^T x)^2 \geq \mathrm{tr}(Xcc^T)\ \forall c \in \mathbb{R}$

The Future

- Experimental results coming soon for problems with duality gap
- Can inexact relaxations provide useful warm starts?

Conclusion

- LP: exact solution, approximate model
- NLP: local solutions to a more accurate model
- Conic: near-optimal solutions to potentially accurate model
- Cuts can improve inexact relaxations

Thank you!