ACOPF: History and Formulations

(Alternating Current Optimal Power Flow)

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- *The author's views do not necessarily represent the views of the Federal Energy Regulatory Commission.*

Outline

- ACOPF background
- Related problems
- Insights from 50 years of history
- ACOPF Formulations

ACOPF: What is it?

- Alternating Current Optimal Power Flow
- Optimization problem – optimize system dispatch subject to system and resource constraints
- Solved in different timeframes
  - Real time market: every 5 minutes
  - Day-ahead market: every 24 hours in hourly increments
  - Capacity market: annually for 3-5 years ahead
  - Transmission planning: annually for 10-15 future years

OPF

- OPF is a general term that describes a class of problems
  - ACOPF Includes
    - full power flow model and
    - system and resource constraints
  - DCOPF Assumes
    - voltage magnitudes constant,
    - voltage angles close to 0,
    - lossless (assume R << X) or lossy system
  - Decoupled OPF
    - Divides the ACOPF into linear subproblems
    - Iterates between the subproblems

ACOPF - basics

- Constraints
  - AC power flow equations
  - Equipment/operating/reliability constraints
    - Voltage, Current, Angle, Real Power, Reactive Power, Apparent Power
- Objective function
  - Maximize social welfare (if demand bids)
  - If demand is fixed, lowest system cost

Related Problems: Power Flow

- Power flow
  - Finds a feasible solution to the power flow equations, but is not an optimization problem
  - Formulated as AC, DC, and decoupled
  - Mismatch
  - Bus type classification
  - Need to match number of variables with number of equations to find solution

Power flow – bus classification

Bus Type	Fixed quantities	Variable quantities	Physical interpretation
PV	Real power (P) Voltage (V)	Reactive power (Q) Angle (θ)	Generator
PQ	Real power (P) Reactive power (Q)	Voltage (V) Angle (θ)	Load, or generator with fixed output
Slack	Voltage (V) Angle (θ)	Real power (P) Reactive power (Q)	An arbitrarily chosen generator

Differences between power flow and OPF

- OPF is an optimization problem with constraints and objective function
  - The number of variables and constraints do not need to match
  - Bus type classifications are unnecessary and may introduce new constraints
- Power flow is a system of equations. It is often solved as a sort of optimization problem with the objective of minimizing “mismatch”

Related Problems: Economic Dispatch

- Economic dispatch
  - Optimization problem – minimize cost subject to generator output limits, overall constraint of generation = load + losses
  - Classic economic dispatch minimizes cost, but does not include network constraints
  - Security-constrained economic dispatch includes network constraints, usually formulated similar to DCOPF or decoupled OPF

History

- Early 20th century:
  - ACOPF ‘solved’ by experienced engineers/operators using judgement, rules of thumb
  - Power flow problem - analog network analyzers
  - Economic dispatch – specialized slide rules

History – mid-century

- 1950's - Digital solutions to the power flow
  - Ward and Hale – 1956
  - Iterative methods based on nodal admittance (Y) or nodal impedance (Z) matrix
  - Gauss-Seidel method
- 1960's – Newton's method for power flow
  - 1960's – Tinney – sparsity techniques

History - 1962

- 50 years ago – Carpentier formulated ACOPF with some key insights:
  - A slack bus unnecessary in an optimization problem
  - Assume problem is suitably convex to apply the KKT conditions
- Based on google scholar, at least 236 papers have cited Carpentier's original 1962 paper, even though it's not available on internet
- 1968 – Dommel and Tinney – reiterate Carpentier's insights, cited by at least 769
- ACOPF formulation has not changed significantly since 1962

History

- Several fairly comprehensive literature review papers since the 1980's
- Stott and Alsac – decoupled OPF
- Literature since 1960's has focused on different solution techniques, modeling improvements
- Challenges of the 1990's persist today
  - Lack of uniform usage or problem definition
  - Local minima (is the problm really suitably convex to apply KKT?)
  - Lack of fast, robust, reliable nonlinear solution algorithms

Notation

- Assumption: balanced 3-phase steady-state operation
- When n and m are subscripts, they index buses;
- k indexes the transmission elements.
- When **j** is not a superscript, $j = (-1)^{1/2}$;
- *i* is the complex current.
- When *j* is a superscript, it is the 'imaginary' part of a complex number.
- Matrices and vectors are upper case.
- Scalars and complex numbers are lower case.

Notation

- For column vectors A and B of length n, where a_k and b_k are the k^{th} components of A and B respectively, the Hadamard product '·' is defined so that $A \cdot B = C$, where C is a column vector also of length n, with k^{th} component $c_k = a_k b_k$.
- The complex conjugate operator is * (superscript) and * (no superscript) is an optimal solution.
- ***Indices and Sets***
- n, m are bus (node) indices; $n, m \in \{1, \ldots, N\}$ where N is the number of buses. (m is an alias for n)
- k is a three-phase transmission element with terminal buses n and m.
- $k \in \{1, \ldots, K\}$ where K is the number of transmission elements between two buses; k counts from 1 to the total number of transmission elements, and does not start over for each bus pair nm.
- K' is the number of a connected bus pairs $(K' \leq K)$.
- Unless otherwise stated, summations ($\sum$) are over the full set of indices.

Notation

- ***Variables***
- p_n is the real power injection (positive) or withdrawal (negative) at bus n
- q_n is the reactive power injection or withdrawal at bus n
- $s_n = p_n + \boldsymbol{j}q_n$ is the net complex power injection at bus n
- p_{nmk} is the real power at bus n to bus m on transmission element k
- q_{nmk} is the reactive power at bus n to bus m on transmission element k
- θ_n is the voltage phase angle at bus n
- $\theta_{nm} = \theta_n - \theta_m$ is the voltage phase angle difference from bus n to bus m

Notation

- ***Variables, continued***
- i is the current (complex phasor); i_n is the current (complex phasor) injection (positive) or withdrawal (negative) at bus n where $i_n = i^r_n + \boldsymbol{j}i^j_n$
- i_{nmk} is the current (complex phasor) injection (positive) or withdrawal (negative) flow in transmission element k at bus n (to bus m). $i_{nmk} = i^r_{nmk} + \boldsymbol{j}i^j_{nmk}$
- s_{nmk} is the apparent complex power injection (positive) or withdrawal (negative) into bus n on transmission element k. $s_{nmk} = s^r_{nmk} + \boldsymbol{j}s^j_{nmk}$
- v_n is the complex voltage at bus n. $v_n = v^r_n + \boldsymbol{j}v^j_n$

Notation

- ***Variables, continued***
- y_{nmk} is the complex admittance on transmission element k connecting bus n and bus m (If buses n and m are not connected directly, $y_{nmk} = 0$.); y_{n0} is the self-admittance (to ground) at bus n.
- y_{nm} is the complex admittance connecting bus n and bus m for all transmission elements k between buses n and m.
- $V = (v_1, \ldots, v_N)^T$ is the complex vector of bus voltages; $V = V^r + \mathbf{j}V^j$
- $I = (i_1, \ldots, i_N)^T$ is the complex vector of bus current injections; $I = I^r + \mathbf{j}I^j$
- $P = (p_1, \ldots, p_N)^T$ is the vector of real power injections
- $Q = (q_1, \ldots, q_N)^T$ is the vector of reactive power injections
- G is the N-by-N conductance matrix
- B is the N-by-N susceptance matrix
- $Y = G + \mathbf{j}B$ is the N-by-N complex admittance matrix

Notation

- **Functions and Transformations**
- *Re( )* is the real part of a complex number, for example, $Re(i^r_n + \boldsymbol{j} i^j_n) = i^r_n$
- *Im( )* is the real part of a complex number, for example, $Im(i^r_n + \boldsymbol{j} i^j_n) = i^j_n$
- | | is the magnitude of a complex number, for example, $|v_n| = [(v^r_n)^2+(v^j_n)^2]^{1/2}$
- *abs( )* is the absolute value function.

Parameters

- **Parameters**
- r_{nmk} is the resistance of transmission element k.
- x_{nmk} is the reactance of transmission element k. s^{max}_k is the thermal limit on apparent power over transmission element k at both terminal buses.
- θ^{min}_{nm}, θ^{max}_{nm} are the maximum and minimum voltage angle differences between n and m
- p^{min}_n, p^{max}_n are the maximum and minimum real power for generator n
- q^{min}_n, q^{max}_n are the maximum and minimum reactive power for generator n
- $C_1 = (c^1_1, \ldots, c^1_N)^T$ *and* $C_2 = (c^2_1, \ldots, c^2_N)^T$ are vectors of linear and quadratic objective function cost coefficients respectively.

Admittance Matrix

- Start with conductance (*G*), susceptance (*B*) and admittance (*Y*) matrices where *gnm, bnm, and ynm* represent elements of the *G*, *B*, and *Y* matrices
- Assume shunt susceptance negligible

$gnmk = rnmk/(rnmk2 + xnmk2)$ for $n \neq m$

$bnmk = -xnmk/(rnmk2 + xnmk2)$ for $n \neq m$

$ynmk = gnmk + \mathbf{j}bnmk$ for $n \neq m$

$ynmk = 0$ for $n = m$

$ynm = -\sum k\ ynmk$ for $n \neq m$

$ynn = yn0 + \sum m\ ynm$

$gnm = -\sum k\ gnmk$ for $n \neq m$

$gnn = gn + \sum m\ gnm$

$bnm = -\sum k\ bnmk$ for $n \neq m$

$bnn = bn + \sum m\ bnm$

Transformers

- Y matrix above does not include transformer parameters.
- For an ideal in-phase transformer (assuming zero resistance in transformer windings, no leakage flux, and no hysteresis loss), the ideal voltage magnitude (turns ratio) is $anmk = |vm|/|vn|$ and $\theta n = \theta m$, where n is the primary side and m is the secondary side of the transformer.
- Since $\theta n = \theta m$, $anmk = |vm|/|vn| = vm/vn = -inm/imn$
- The current-voltage equations for ideal transformer *k* between buses *n* and *m* are:
- $inmk = anmk2ynmkvn - anmkynmkvm$
- $imnk = -anmkynmkvn + ynmkvm$
- For the phase shifting transformer (PAR) with a phase angle shift of φ,
- $vm/vn = tnmk = anmkej\varphi$
- $inm/imn = tnmk^* = -anmke\text{-}j\varphi$
- The current-voltage (IV) equations for the phase shifting transformer *k* between buses *n* and *m* are:
- $inmk = anmk2ynmkvn - tnmk^*ynmkvm$
- $imnk = -tnmkynmkvn + ynmkvm$

Circuit Equations

- Kirchhoff's current law requires that the sum of the currents injected and withdrawn at bus n equal zero:
- $i_n = \sum_k i_{nmk}$ (2)
- If we define current to ground to be $y_{n0}(v_n - v_0)$ and $v_0 = 0$, we have:
- $i_n = \sum_k y_{nmk}(v_n - v_m) + y_{n0}v_n$ (6)
- $i_{nmk} = y_{nmk}(v_n - v_m) =$
  $g_{nmk}(v^r_n - v^r_m) - b_{nmk}(v^j_n - v^j_m) + \mathbf{j}(b_{nmk}(v^r_n - v^r_m) + g_{nmk}(v^j_n - v^j_m))$
- $i^r_{nmk} = g_{nmk}(v^r_n - v^r_m) - b_{nmk}(v^j_n - v^j_m)$
- $i^j_{nmk} = b_{nmk}(v^r_n - v^r_m) + g_{nmk}(v^j_n - v^j_m)$
- Current is a linear function of voltage. Rearranging,
- $i_n = v_n(y_{n0} + \sum_k y_{nmk}) - \sum_k y_{nmk}v_m$ (8)

AC Power Flow Equations

- In matrix notation, the IV flow equations in terms of current (I) and voltage (V) in (8) are
- $I = YV = (G + \mathbf{j}B)(V^r + \mathbf{j}V^j) = GV^r - BV^j + \mathbf{j}(BV^r + GV^j)$ (12)
- where $I^r = GV^r - BV^j$ and $I^j = BV^r + GV^j$
- *In I and V, the flow equations are linear*

In another matrix format, (12) is

$I = (I^r, I^j) = \underline{Y}(V^r, V^j)^T$ or

$$I = (I^r, I^j) = \begin{bmatrix} G & -B \\ B & G \end{bmatrix} \begin{bmatrix} V^r \\ V^j \end{bmatrix} \quad \text{where } \underline{Y} = \begin{bmatrix} G & -B \\ B & G \end{bmatrix}$$

Power Flow Equations

- The traditional power-voltage power flow equations defined in terms of real power (P), reactive power (Q) and voltage (V) are
- $S = P + \boldsymbol{j}Q = diag(V)I^* = diag(V)[YV]^* = diag(V)Y^*V^*$ (16)
- The power injections are
- $S = V \bullet I^* = (V^r + \boldsymbol{j}V^j) \bullet (I^r - \boldsymbol{j}I^j) = (V^r \bullet I^r + V^j \bullet I^j) + \boldsymbol{j}(V^j \bullet I^r - V^r \bullet I^j)$ (18)
- where
- $P = V^r \bullet I^r + V^j \bullet I^j$ (20)
- $Q = V^j \bullet I^r - V^r \bullet I^j$ (22)
- The power-voltage power flow equations (16) and (18) are quadratic. The IV flow equations (12) are linear.

Constraints

- Generator and load constraints
  - $P^{min} \le P \le P^{max}$ $\qquad$ $Q^{min} \le Q \le Q^{max}$
  - In terms of V and I, the injection constraints are:
  - $P^{min} \le V^r \bullet I^r + V^j \bullet I^j \le P^{max}$
  - $Q^{min} \le V^j \bullet I^r - V^r \bullet I^j \le Q^{max}$
- Voltage constraints
- $(v^{min}{}_m)^2 \le (v^r{}_m)^2 + (v^j{}_m)^2 \le (v^{max}{}_m)^2$
- In matrix form: $(V^{min})^2 \le V^r \bullet V^r + V^j \bullet V^j \le (V^{max})^2$
- Line flow thermal constraints
- The apparent power at bus *n* on transmission element *k* to bus *m* is: $s_{nmk} = v_n i_{nmk}{}^* = v_n y_{nmk}(v_n - v_m). = v_n y_{nmk} v_n - v_n y_{nmk} v_m).$
- The thermal limit on s_{nmk} is: $(s^r{}_{nmk})^2 + (s^j{}_{nmk})^2 = |s_{nmk}|^2 \le (s^{max}{}_k)^2$
- Or The thermal limit on i_{nmk} is: $(i^r{}_{nmk})^2 + (i^j{}_{nmk})^2 = |i_{nmk}|^2 \le (i^{max}{}_k)^2$

Objective Functions

- The economically efficient objective function is to maximize social welfare. In the case of the OPF with fixed demand, that is the same as minimizing system cost.
  - Areas to explore – adding cost of reactive power, adding cost of switching
- Others:
  - Minimize losses
  - Minimize fuel cost
  - Minimize emissions
  - Minimize control actions
  - All of these other objective functions are redundant or sub-optimal in a ACOPF that models constraints and costs.

ACOPF Formulations

- Three formulations:
  - Polar P-Q (most common in literature)
  - Rectangular P-Q (less common in literature)
  - Rectangular I-V (new)
  - There are also a variety of hybrid formulations.

Formulations: Polar P-Q

- Network-wide objective function: *Min c(S)* (40')
- Network-wide constraints:
- $P_n = \sum_{mk} V_n V_m (G_{nmk} \cos\theta_{nm} + B_{nmk} \sin\theta_{nm})$ (41')
- $Q_n = \sum_{mk} V_n V_m (G_{nmk} \sin\theta_{nm} - B_{nmk} \cos\theta_{nm})$ *(41')*
  - These are quadratic-trigonometric equalities

- $V^{min} \leq V \leq V^{max}$ (46'-47')
- $\theta^{min}_{nm} \leq \theta_n - \theta_m \leq \theta^{max}_{nm}$ (49')

Formulations: Rectangular P-Q

- Network-wide objective function: *Min c(S)*
- Network-wide constraint: $P + \boldsymbol{j}Q = S = V \bullet I^* = V \bullet Y^* V^*$ (41)
- Bus-specific constraints
- $P^{min} \leq P \leq P^{max}$ (43)
- $Q^{min} \leq Q \leq Q^{max}$ (45)
- (46')-(47') are replaced by:
- $V^r \bullet V^r + V^j \bullet V^j \leq (V^{max})^2$ (46)
- $(V^{min})^2 \leq V^r \bullet V^r + V^j \bullet V^j$ (47)
- $(|s_{nmk}|)^2 \leq (s^{max}{}_k)^2$ for all k (48)
- (49') is replaced by:
- $\theta^{min}{}_{nm} \leq \arctan(v^j{}_n / v^r{}_n) - \arctan(v^j{}_m / v^r{}_m) \leq \theta^{max}{}_{nm}$ (49)
- $V^r \geq 0$ (49.1)

Formulations: Rectangular I-V

- Network-wide objective function: *Min c(S)* (50)
- Network-wide constraint: $I = YV$ (51)
- Bus-specific constraints:
- $P = V^r \bullet I^r + V^j \bullet I^j \leq P^{max}$ (54)
- $r \leq P = V^r \bullet I^r + V^j \bullet I^j$ (55)
- $Q = V^j \bullet I^r - V^r \bullet I^j \leq Q^{max}$ (56)
- $Q^{min} \leq Q = V^j \bullet I^r - V^r \bullet I^j$ (57)
- $V^r \bullet V^r + V^j \bullet V^j \leq (V^{max})^2$ (58)
- $(V^{min})^2 \leq V^r \bullet V^r + V^j \bullet V^j$ (59)
- $(i_{nmk})^2 \leq (i^{max}{}_k)^2$ for all k (60)
- $\theta^{min}{}_{nm} \leq \arctan(v^j{}_n/v^r{}_n) - \arctan(v^j{}_m/v^r{}_m) \leq \theta^{max}{}_{nm}$ (61)
- Can (60) make (61) redundant?
- $V^r \geq 0$ (62)

Comparison of Formulations

Formulation	**Polar PQ**	**Rectangular PQ**	**Rectangular IV**
Network constraints	2N nonlinear quadratic and trigonometric equality constraints	2N quadratic equalities	2N **linear** equality constraints
Angle difference constraints	**Linear**	Nonconvex (arctan); **Linear** if replaced with current or apparent power constraint	Nonconvex (arctan); **Linear** if replaced with current or apparent power constraint
Bus constraints	**Linear**	Noncovex quadratic inequalities	Locally quadratic, some nonconvex, some convex

Conclusions

- The ACOPF problem is inherently difficult due to nonconvexities, multipart nonlinear pricing, and alternating current.
- We do not yet have practical approaches to solving nonconvex problems.
- The ACOPF is a well-structured problem, and has developed during 50 years of research.
- The ACOPF is not a hypothetical problem – it is solved every 5 minutes through approximations and judgment.
- People have researched the ACOPF for 50 years, but there are still a lot of possibilities and ways to examine it.
- There is not yet a commercially viable full ACOPF. Since today's solvers do not return the gap between the given and globally optimal solution.
- If we make a rough estimate that today's solvers are on average off by 10%, and world energy costs are $400 billion, closing the gap by 10% is a huge financial impact.

Thank You

- Questions?

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