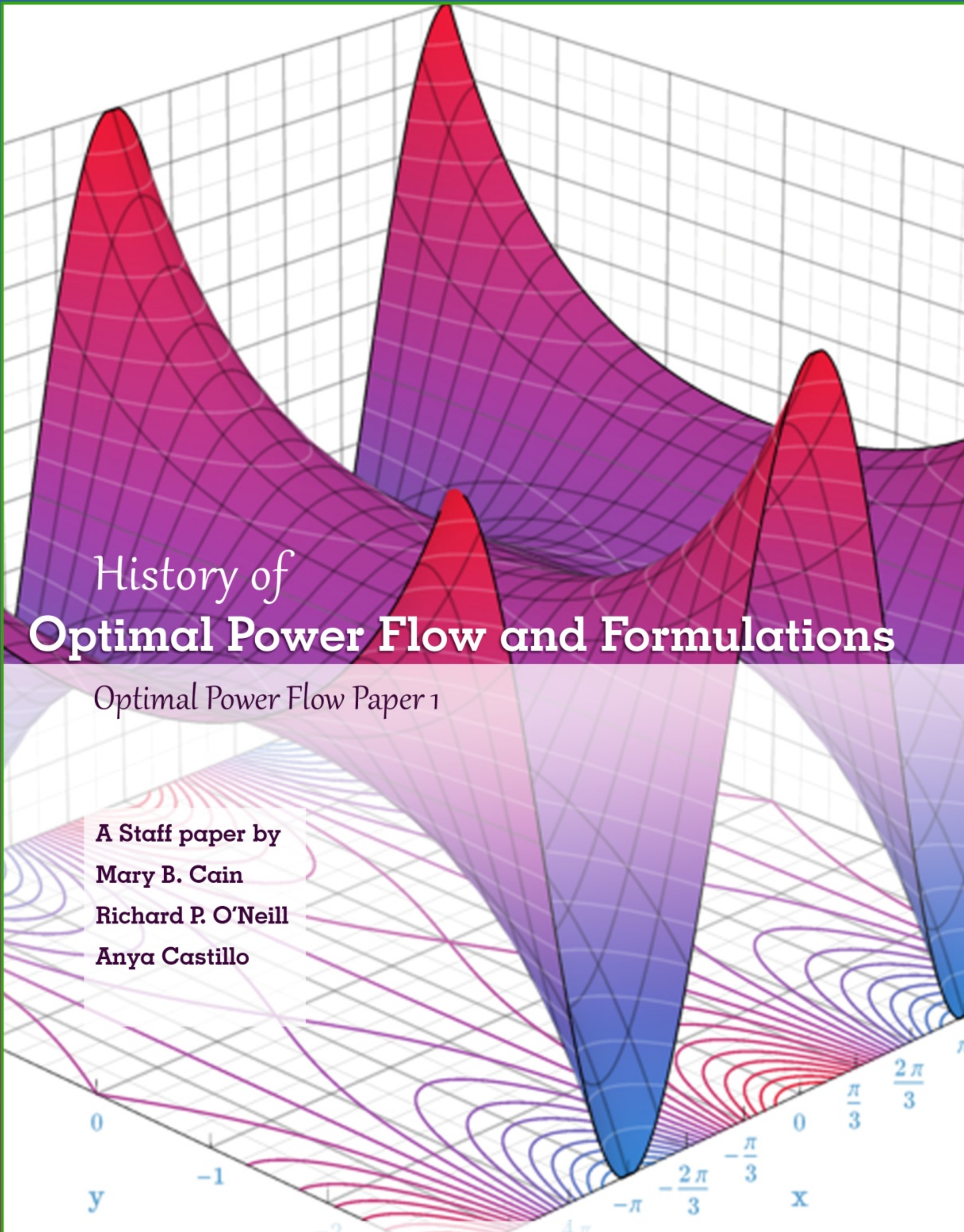

History of

Optimal Power Flow and Formulations

Optimal Power Flow Paper 1

A Staff paper by

Mary B. Cain

Richard P. O'Neill

Anya Castillo

DECEMBER 2012

History of Optimal Power Flow and Formulations

Optimal Power Flow Paper 1

Mary B. Cain, Richard P. O'Neill, Anya Castillo

mary.cain@ferc.gov; richard.oneill@ferc.gov; anya.castillo@ferc.gov

December, 2012

Abstract:

The purpose of this paper is to present a literature review of the AC Optimal Power Flow (ACOPF) problem and propose areas where the ACOPF could be improved. The ACOPF is at the heart of Independent System Operator (ISO) power markets, and is solved in some form every year for system planning, every day for day-ahead markets, every hour, and even every 5 minutes. It was first formulated in 1962, and formulations have changed little over the years. With advances in computing power and solution algorithms, we can model more of the constraints and remove unnecessary limits and approximations that were previously required to find a solution in reasonable time. One example is nonlinear voltage magnitude constraints that are modeled as linear thermal proxy constraints. In this paper, we refer to the full ACOPF as an ACOPF that simultaneously optimizes real and reactive power. Today, 50 years after the problem was formulated, we still do not have a fast, robust solution technique for the full ACOPF. Finding a good solution technique for the full ACOPF could potentially save tens of billions of dollars annually. Based on our literature review, we find that the ACOPF research community lacks a common understanding of the problem, its formulation, and objective functions. However, we do not claim that this literature review is a complete review—our intent was simply to capture the major formulations of the ACOPF. Instead, in this paper, we seek to clearly present the ACOPF problem through clear formulations of the problem and its parameters. This paper defines and discusses the polar *power-voltage*, rectangular *power-voltage*, and rectangular *current-voltag*e formulations of the ACOPF. Additionally, it discusses the different types of constraints and objective functions. This paper lays the groundwork for further research on the convex approximation of the ACOPF solution space, a survey of solution techniques, and computational performance of different formulations.

Table of Contents

1. Introduction

The heart of economically efficient and reliable Independent System Operator (ISO) power markets is the alternating current optimal power flow (ACOPF) problem. This problem is complex economically, electrically and computationally. Economically, an efficient market equilibrium requires multi-part nonlinear pricing. Electrically, the power flow is alternating current (AC), which introduces additional nonlinearities. Computationally, the optimization has nonconvexities, including both binary variables and continuous functions, which makes the problem difficult to solve. The power system must be able to withstand the loss of any generator or transmission element, and the system operator must make binary decisions to start up and shut down generation and transmission assets in response to system events. For investment planning purposes, the problem needs binary investment variables and a multiple year horizon.

Even 50 years after the problem was first formulated, we still lack a fast and robust solution technique for the full ACOPF. We use approximations, decompositions and engineering judgment to obtain reasonably acceptable solutions to this problem. While superior to their predecessors, today's approximate-solution techniques may unnecessarily cost tens of billions of dollars per year. They may also result in environmental harm from unnecessary emissions and wasted energy. Using EIA data on wholesale electricity prices and U.S. and World energy production, Table 1 gives a range of potential cost savings from a 5% increase in market efficiency due to improvements to the ACOPF.(EIA 2012). Small increases in efficiency of dispatch are measured in billions of dollars per year. Since the usual cost of purchasing and installing new software for an existing ISO market is less than $10 million dollars (O'Neill et. al. 2011), the potential benefit/cost ratios of better software are in the range of 10 to 1000.

TABLE 1: POTENTIAL COST SAVINGS OF INCREASED EFFICIENCY OF DISPATCH (EIA 2012)

	2009 gross electricity production (MWh)	Production cost ($billion/year) assuming $30/MWh energy price	Savings ($billion/year) assuming 5% increase in efficiency	Production cost ($billion/year) assuming $100/MWh energy price	Savings ($billion/year) assuming 5% increase in efficiency
U.S.	3,724,000	112	6	372	19
World	17,314,000	519	26	1731	87

An ultimate goal of ISO market software, and a topic of future research, is the security-constrained, self-healing (corrective switching) AC optimal power flow with unit commitment over the optimal network. The optimal network is flexible, with assets that have time-varying dynamic ratings reflecting the asset capability under varying operating conditions. The optimal network is also optimally configured – opening or closing transmission lines becomes a decision variable, or control action, rather than an input to the problem, or state. When possible, the security constraints are corrective rather than preventive. With preventive security constraints, the system is operated conservatively to survive loss of any transmission element or generator. In contrast, corrective constraints reconfigure the system with fast-acting equipment such as special protection systems or remedial action schemes immediately following loss of a generator or transmission element, allowing the system to be reliably used closer to its limits. This problem must be solved weekly in 8 hours, daily in 2 hours, hourly in 15 minutes, each five minutes in 1 minute and for self-healing post-contingency in 30 seconds. Currently, the problem is solved through varying levels of approximation, depending on application and time scale, but with increases in computing power it may be possible to reduce the number of approximations and take advantage of parallel computing.

Today, the computational challenge is to consistently find a global optimal solution with speeds up to three to five orders of magnitude faster than existing solvers. There is some promising recent evidence that this could be a reality in five to ten years. For example, in the last two decades mixed-integer programming (MIP) has achieved speed improvements of 10^7; that is, problems that would have taken

10 years in 1990 can be solved in one minute today. As a consequence, MIP is replacing other approaches in ISO markets. Implementation of MIP into the day-ahead and real-time markets, with the Commission's encouragement, has saved American electricity market participants over one-half billion dollars per year (FERC 2011). More will be saved as all ISOs implement MIP and the new formulations it permits in the next several years.

Due to idiosyncrasies in design, current software oversimplifies the problem in different ways, and requires operator intervention to address real-time problems that do not show up in models. This operator intervention unnecessarily alters settlement prices and produces suboptimal solutions. The Joint Board on Economic Dispatch for the Northeast Region stated in 2006 that improved modeling of system constraints such as voltage and stability constraints would result in more precise dispatches and better market signals, but that the switch to AC-based software would increase the time to run a single scenario from minutes to over an hour, making use of ACOPF impractical, even for the day-ahead market (FERC 2006). One example is the Midwest Independent System Operator (MISO), where operators have to commit resources before the unit commitment and economic dispatch software models are run to address local voltage issues that MISO has had difficulty modeling in its market software (FERC 2012). PJM Interconnection (PJM) employs an approach, called Perfect Dispatch, that ex-post solves the real-time market problem with perfect information (PJM 2012). The Perfect Dispatch solution is used to train operators, where they can compare the "perfect dispatch," which is based on "perfect" after-the-fact information to the actual dispatch, which is based on the information available at the time. ISO models solve proxies or estimates for reactive power and voltage constraints, where they calculate linear thermal constraints to approximate quadratic voltage magnitude constraints. The details of transmission constraint modeling and transmission pricing have been neglected, but need to be considered to improve the accuracy of ACOPF calculations. Transmission constraints can be modeled in terms of current, real power, apparent power, voltage magnitude differences, or angle differences. The choice of constraint depends on the type of model, data availability, and physical limit (voltage, stability, or thermal limit). Surrogate constraints can be calculated based on the line flow equations, but these calculations have inherent assumptions. One example is the Arizona-Southern California outage in 2011, where some line limits were modeled and monitored as real power transfer limits while others were modeled as current transfer limits (FERC/NERC 2012). This paper seeks to better understand the ACOPF problem through clear formulations of the problem, theoretical properties of the problem and its parameters, approximations to the nonlinear functions that are necessary to make the problem solvable, and to produce computational results from large and small test problems using various solvers and starting points. Discrete variables such as equipment states, generator commitments, and transmission switching further complicate the ACOPF, but we do not discuss these in this paper. With the increased measurements and controls inherent in smart grid upgrades, the potential savings are greater, although the problem may become more complex with more discrete devices to model.

In the rest of the paper, we provide a brief history of power system optimization, present notation and nomenclature, formulate the admittance matrix and power flow equations, formulate constraints, present different formulations of the ACOPF, and present a literature review of ACOPF formulations.

2. History of Power System Optimization

Power system optimization has evolved with developments in computing and optimization theory. In the first half of the 20th century, the optimal power flow problem was "solved" by experienced engineers and operators using judgment, rules of thumb, and primitive tools, including analog network analyzers and specialized slide rules. Gradually, computational aids were introduced to assist the intuition of operator experience. The optimal power flow problem was first formulated in the 1960's (Carpentier 1962), but has proven to be a very difficult problem to solve. Linear solvers are widely available for linearized versions of the optimal power flow problem, but nonlinear solvers cannot guarantee a global optimum, are not robust, and do not solve fast enough. In each electricity control room, the optimal power flow problem or an approximation must be solved many times a day, as often as every 5 minutes.

There are three types of problems commonly referred to in power system literature: power flow (load flow), economic dispatch, and optimal power flow. Three other classes of power system optimization, specifically unit commitment, optimal topology, and long-term planning, involve binary and integer variables, and are not discussed in this paper; but combined with the insights on formulations in this paper, could be promising areas for future research.

Table 2 compares the major characteristics of the power flow, economic dispatch, and optimal power flow problems. The power flow or load flow refers to the generation, load, and transmission network equations. Power flow methods find a mathematically but not necessarily physically feasible or optimal solution. The power flow equations themselves do not take account of limitations on generator reactive power limits or transmission line limits, but these constraints can be programmed into many power flow solvers.

A second type of problem, economic dispatch, describes a variety of formulations to determine the least-cost generation dispatch to serve a given load with a reserve margin, but these formulations simplify or sometimes altogether ignore power flow constraints.

A third type of problem, the optimal power flow, finds the optimal solution to an objective function subject to the power flow constraints and other operational constraints, such as generator minimum output constraints, transmission stability and voltage constraints, and limits on switching mechanical equipment. Optimal power flow is sometimes referred to as security-constrained economic dispatch (SCED); most implementations of SCED include only thermal limits, and proxies for voltage limits. There are a variety of formulations with different constraints, different objective functions, and different solution methods that have been labeled optimal power flow; these are discussed in the formulations section later in this paper. Formulations that use the exact AC power flow equations are known as "ACOPF." Simpler versions, known as DCOPF, assume all voltage magnitudes are fixed and all voltage angles are close to zero. DC stands for direct current, but is a bit of a misnomer; a DCOPF is a linearized form of a full alternating current network (ACOPF) and not a power flow solution for a direct current network. We use the general term OPF to include both ACOPF and DCOPF. The ACOPF is often solved through decoupling, which takes advantage of the structure of the problem, where real power (P) and voltage angle (θ) are tightly coupled and voltage magnitude (V) and reactive power (Q) are tightly coupled, but the P-θ and V-Q problems are weakly coupled due to the assumptions that the phase angle differences between adjacent buses are rather small, and high-voltage transmission networks have much higher reactance compared to resistance. The decoupled OPF divides the ACOPF into two linear subproblems, one with power and voltage angle and another with voltage magnitude and reactive power. In this paper, we use the term ACOPF to refer to the full ACOPF that simultaneously optimizes real and reactive power, and decoupled OPF to refer to the decoupled problems that separately optimize real and reactive power and iterate between the two to reach an optimal solution.

TABLE 2: MAJOR TYPES OF POWER SYSTEM PROBLEMS

General problem type	Problem name	Includes voltage angle constraints?	Includes bus voltage magnitude constraints?	Includes transmission constraints?	Includes losses?	Assumptions	Includes generator costs?	Includes contingency constraints?
OPF	ACOPF, or Full ACOPF	Yes	Yes	Yes	Yes		Yes	No
OPF	DCOPF	No	No; all voltage magnitudes fixed	Yes	Maybe	Voltage magnitudes are constant	Yes	No
OPF	Decoupled OPF	Yes	Yes	Yes	Yes	Power-voltage angle are independent of voltage magnitude-reactive power	Yes	No
OPF	Security-Constrained Economic Dispatch (SCED)	Yes	No	Yes	Yes	Voltage magnitudes are constant	Yes	Yes
Power flow	Power Flow, or Load Flow	No, but can be added	Yes	No, but can be added	Yes		No	No
Economic dispatch	Economic Dispatch	No	No	No	Depends	No transmission constraints	Yes	No
OPF	Security Constrained OPF (SCOPF)	Yes	Depends	Yes	Yes	Depends	Yes	Yes

We now discuss early research of the three types of problems in power system optimization: economic dispatch, power flow, and optimal power flow.

As early as the 1930's, the economic dispatch problem was solved by hand or specially-developed slide rule using the principle of equal incremental loading, taking as long as 8 hours to complete (Happ 1977). Early computations of economic dispatch were slow. Kirchmayer estimated that it would take 10 minutes of computational time to produce the schedules for a 10 generator system at a given system price (Kirchmayer 1958). In contrast, RTOs today solve systems of hundreds of generators in a matter of seconds. In the survey of economic dispatch methods up through the 1970's, Happ provides an overview of the evolution of economic dispatch formulations and different ways to account for losses.

Prior to digital computers, as early as 1929, the power flow problem was solved with analog network analyzers that simulated power systems (Sasson 1967). Ward and Hale published the first automated digital solution to the power flow problem in 1956 (Ward 1956). Sasson and Jaimes provide a survey and comparison of early load flow solution methods, which are various iterative methods based on the nodal admittance matrix (Y matrix) or its inverse, the nodal impedance matrix (Z matrix) (Sasson 1967). Early researchers, including Carpentier, used the Gauss-Seidel method. The Newton-Raphson method became the commonly used solution method during the 1960's (Peschon et. al. 1968), after Tinney and others developed sparsity techniques to take advantage of the structure of the admittance matrix in the OPF problem. The admittance matrix is sparse, meaning it has many zero elements; this is because power system networks are not densely connected. Sparsity techniques can be used to reduce data storage and increase computation speed (Stott 1974).

Early research on OPF used classical Lagrangian techniques for the optimality conditions, but neglected bounds on variables (Squires 1961). In 1962, Carpentier published the optimality conditions for an OPF, including variable bounds, based on the Kuhn-Tucker conditions; this is generally considered the first publication of a fully formulated OPF (Carpentier 1962). Carpentier assumes that the applicable functions display "suitable convexity" for the Kuhn-Tucker (now referred to as the Karush-Kuh-Tucker or KKT) conditions to apply (Carpentier 1962). Given the structure of the power flow equations, this may be a big assumption (Hiskens 2001 and Schecter 2012). Carpentier includes the full AC power flow equations, generator real and reactive power constraints, bus voltage magnitude constraints, and bus voltage angle difference constraints for buses connected by transmission elements.

Huneault and Galliana provide an extensive survey of optimal power flow literature up to 1991, surveying over 300 articles and citing 163 (Huneault 1991). They conclude, "The history of optimal power flow (OPF) research can be characterized as the application of increasingly powerful optimization tools to a problem which basically has been well-defined since the early 1960's." The paper outlines the evolution of OPF literature, grouped by solution method. The solution methods include various forms of gradient methods, linear programming, quadratic programming, and penalty methods. The authors conclude that "commercially available OPF algorithms all satisfy the full nonlinear load flow model and a full set of bounds on variables." The authors further conclude that the OPF remains a difficult mathematical problem. The present algorithms cannot compute quickly enough, and are prone to serious ill-conditioning and convergence problems.

Another area of research, security-constrained OPF, accounts for transmission contingency constraints and poses additional computational challenges (Carpentier 1979, Stott 1987). Our discussion in this paper focuses on ACOPF. Future research could extend the formulations to include contingency constraints that are required to maintain the system after an outage. This formulation increases the size of the problem formulation by a factor equal to the number of contingencies studied.

Researchers have identified challenges to solving the OPF, including modeling discrete variables, local minima, lack of uniform problem definition, solution reliability and computing time. Some of these have been solved: both Tinney et al. and Momoh et al. discussed the challenges in modeling discrete variables in OPF solutions (Tinney 1988), (Momoh 1997). Today, with advances in mixed integer programming (MIP), discrete variables can be modeled and included in ACOPF solutions. Other challenges persist today: Koessler states that the "lack of uniformity in usage and definition" has been a challenge to users and developers in OPF, and specifically discusses local minima, which indicate that the problem is nonconvex (Momoh 1997). Huneault and Galliana found that algorithms available in 1991 could not compute OPF solutions quickly and reliably enough, and that the OPF, like many nonlinear problems, is prone to ill-conditioning and difficult convergence (Huneault 1991).

3. Conventions, Parameters, Sets and Variables

Notation and Nomenclature

When n and m are subscripts, they index buses; k indexes the transmission elements. When $\boldsymbol{j}$ is not a superscript, $\boldsymbol{j} = (-1)^{1/2}$; i is the complex current. When j is a superscript, it is the 'imaginary' part of a complex number. Matrices and vectors are represented with upper case letters. Scalars and complex numbers are in lower case letters. For column vectors A and B of length n, where a_k and b_k are the k^{th} components of A and B respectively, the Hadamard product '$\cdot$' is defined so that $A \cdot B = C$, where C is a column vector also of length n, with k^{th} component $c_k = a_k b_k$.

The complex conjugate operator is * (superscript) and * (no superscript) is an optimal solution.

We assume balanced, three-phase, steady-state conditions. All variables are associated with a single-line model of a balanced, three-phase system. A common practice in power system modeling is the per-unit (p.u.) representation, where base quantities for voltage magnitude, current, power, and impedance (or admittance) are used to normalize quantities in a network with multiple voltage levels. Such normalization is a convenience. We use the convention that an injection occurs when the real part of the complex number is positive and a withdrawal occurs when the real part of the complex number is negative.

The topology of the network consists of locations known as buses or nodes and transmission elements connecting paired buses. The network is an undirected planar graph.

Indices and Sets

n, m are bus (node) indices; $n, m \in \{1, \ldots, N\}$ where N is the number of buses. (m is an alias for n)

k is a three-phase transmission element with terminal buses n and m.

$k \in \{1, \ldots, K\}$ where K is the number of transmission elements; k counts from 1 to the total number of transmission elements, and does not start over for each bus pair nm.

K' is the set of connected bus pairs nm ($|K'| \leq |K|$).

Unless otherwise stated, summations ($\sum$) are over the full set of indices.

Variables

p_n is the real power injection (positive) or withdrawal (negative) at bus n

q_n is the reactive power injection or withdrawal at bus n

$s_n = p_n + \boldsymbol{j}q_n$ is the net complex power injection or withdrawal at bus n

We distinguish between the real, reactive, or complex power injected at a specific bus (p_n, q_n, *and* s_n) and the real, reactive, or complex power flowing in a transmission element between two buses:

p_{nmk} is the real power flow from bus n to bus m on transmission element k

q_{nmk} is the reactive power flow from bus n to bus m on transmission element k

s_{nmk} is the apparent complex power flow from bus n on transmission element k. $s_{nmk} = s^r_{nmk} + \boldsymbol{j}s^j_{nmk} = p_{nmk} + \boldsymbol{j}\, q_{nmk}$

θ_n is the voltage angle at bus n

$\theta_{nm} = \theta_n - \theta_m$ is the voltage angle difference from bus n to bus m

$\theta - \delta$ is the power angle.

i is the current (complex phasor); we distinguish between current injected at a specific bus and current flowing in a transmission element between two buses:

i_n is the current (complex phasor) injection (positive) or withdrawal (negative) at bus n where $i_n = i^r_n + \boldsymbol{j}i^j_n$

i_{nmk} is the current (complex phasor) flow in transmission element k at bus n (to bus m). $i_{nmk} = i^r_{nmk} + \boldsymbol{j}i^j_{nmk}$

v_n is the complex voltage at bus n. $v_n = v^r_n + \boldsymbol{j}v^j_n$

y_{nmk} is the complex admittance on transmission element k connecting bus n and bus m (If buses n and m are not connected directly, $y_{nmk}= 0$.); y_{n0} is the self-admittance (to ground) at bus n.

$V = (v_1, \ldots, v_N)^T$ is the complex vector of bus voltages; $V = V^r + \mathbf{j}V^j$

$I = (i_1, \ldots, i_N)^T$ is the complex vector of bus current injections; $I = I^r + \mathbf{j}I^j$

$P = (p_1, \ldots, p_N)^T$ is the vector of real power injections

$Q = (q_1, \ldots, q_N)^T$ is the vector of reactive power injections

G is the N-by-N conductance matrix

B is the N-by-N susceptance matrix

Note that elements of G and B will be constant for passive transmission elements such as transmission lines, but can be variable when active transmission elements such as phase shifting transformers, switched capacitors/reactors, or power electronic flexible AC transmission system (FACTS) devices are included.

$Y = G + \mathbf{j}B$ is the N-by-N complex admittance matrix

g_{nm}, b_{nm}, and y_{nm} represent elements of the G, B, and Y matrices respectively.

Functions and Transformations

$Re(\ )$ is the real part of a complex number, for example, $Re(i^r_n + \mathbf{j}i^j_n) = i^r_n$

$Im(\ )$ is the real part of a complex number, for example, $Im(i^r_n + \mathbf{j}i^j_n) = i^j_n$

$|\ |$ is the magnitude of a complex number, for example, $|v_n| = [(v^r_n)^2+(v^j_n)^2]^{1/2}$

$abs(\ )$ is the absolute value function.

The transformation from rectangular to polar coordinates for complex voltage is:

$$v^r_n = |v_n|cos(\theta_n)$$

$$v^j_n = |v_n|sin(\theta_n)$$

$$(v^r_n)^2 + (v^j_n)^2 = [|v_n|sin(\theta_n)]^2 + [|v_n|cos(\theta_n)]^2 = |v_n|^2[sin(\theta_n)^2 + cos(\theta_n)^2] = |v_n|^2$$

We drop the bus index n and let θ be the voltage angle and δ be the current angle.

For real power,

$$p = v^r i^r + v^j i^j = |v|cos\theta|i|cos\delta + |v|sin\theta|i|sin\delta = |v||i|[cos\theta cos\delta + sin\theta sin\delta]$$

$$= |v||i|(0.5[cos(\theta-\delta)+ cos(\theta+\delta)] + 0.5[cos(\theta-\delta)- cos(\theta+\delta)]\ )$$

$$= |v||i|cos(\theta-\delta)$$

For reactive power,

$$q = v^j i^r - v^r i^j = |v|sin\theta|i|cos\delta - |v|cos\theta|i|sin\delta = |v||i|[sin\theta cos\delta - cos\theta sin\delta]$$

$= |v||i|.5[sin(\theta + \delta) + sin(\theta - \delta)] - |v||i|.5[sin(\theta + \delta) - sin(\theta - \delta)]$

$q = |v||i|sin(\theta - \delta)$

$\theta - \delta$ is the power angle.

Parameters

r_{nmk} or r_k is the resistance of transmission element k.

x_{nmk} or x_k is the reactance of transmission element k.

$s^{max}{}_k$ is the thermal limit on apparent power over transmission element k at both terminal buses.

$\theta^{min}{}_{nm}$, $\theta^{max}{}_{nm}$ are the maximum and minimum voltage angle differences between n and m

$p^{min}{}_n$, $p^{max}{}_n$ are the maximum and minimum real power for generator n

$q^{min}{}_n$, $q^{max}{}_n$ are the maximum and minimum reactive power for generator n

$C_1 = (c^1{}_1, \ldots, c^1{}_N)^T$ *and* $C_2 = (c^2{}_1, \ldots, c^2{}_N)^T$ are vectors of linear and quadratic objective function cost coefficients respectively.

4. Admittance Matrix and AC Power Flow Equations

In this section, we develop the admittance matrix and the *current-voltage* flow equations (IV equations), which are a different formulation of the commonly used power flow equations. In the following sections, we develop the additional constraints that bound the solutions.

We define the conductance (G), susceptance (B) and admittance (Y) matrices, with elements g_{nm}, b_{nm}, and y_{nm} respectively, and $Y = G + \boldsymbol{j}B$. We start with a simple admittance matrix defined by resistance, r, and reactance, x. We assume shunt susceptance is negligible. The elements of G, B and Y matrices are derived as follows:

$g_{nmk} = r_{nmk}/(r_{nmk}^2 + x_{nmk}^2)$ for $n \neq m$

$b_{nmk} = -x_{nmk}/(r_{nmk}^2 + x_{nmk}^2)$ for $n \neq m$

$y_{nmk} = g_{nmk} + \boldsymbol{j}b_{nmk}$ for $n \neq m$

$y_{nmk} = 0$ for $n = m$

$y_{nm} = \sum_k y_{nmk}$ for $n \neq m$

$y_{nn} = y_{n0} - \sum_{n \neq m} y_{nm}$

Transformers. The admittance matrix above does not include transformer parameters. For an ideal in-phase transformer (assuming zero resistance in transformer windings, no leakage flux, and no hysteresis loss), the ideal voltage magnitude (turns ratio) is $a_{nmk} = |v_m|/|v_n|$ *and* $\theta_n = \theta_m$, where n is the primary side and m is the secondary side of the transformer. Since $\theta_n = \theta_m$,

$$a_{nmk} = |v_m|/|v_n| = v_m/v_n = -i_{nm}/i_{mn}$$

The *current-voltage* (IV) equations for ideal transformer *k* between buses *n* and *m* are:

$$i_{nmk} = a_{nmk}^2 y_{nmk} v_n - a_{nmk} y_{nmk} v_m$$

$$i_{mnk} = -a_{nmk} y_{nmk} v_n + y_{nmk} v_m$$

For the phase shifting transformer (PAR) with a phase angle shift of φ,

$$v_m/v_n = t_{nmk} = a_{nmk} e^{j\varphi}$$

$$i_{nm}/i_{mn} = t_{nmk}^* = -a_{nmk} e^{-j\varphi}$$

The *current-voltage* (IV) equations for the phase shifting transformer *k* between buses *n* and *m* are:

$$i_{nmk} = a_{nmk}^2 y_{nmk} v_n - t_{nmk}^* y_{nmk} v_m$$

$$i_{mnk} = -t_{nmk} y_{nmk} v_n + y_{nmk} v_m$$

Admittance Matrix. If there are no transformers or FACTS devices, *G* is positive semidefinite and *B* is negative semidefinite. A matrix where $y_{nn} \geq abs(\sum_m y_{nm})$ is called diagonally dominant and strictly diagonally dominant if $y_{nn} > abs(\sum_m y_{nm})$.

If there are no transformers and $y_{n0} = 0$, *G* and *B* are weighted Laplacian matrices of the undirected weighted graph that describes the transmission network. Much is known about the weighted Laplacian matrices. *Y* is a complex weighted Laplacian matrix. The admittance matrix is $Y = G + \boldsymbol{j}B$, where *G* and *B* are real symmetric diagonally dominant matrices. A symmetric diagonally dominant matrix has a symmetric factorization, for example, $B = UU^T$ where each column of *U* has at most two non-zeros and the non-zeroes have the same absolute value.

For large problems, the admittance matrix, $Y = G + \boldsymbol{j}B$, is usually sparse. The density of both *G* and *B* is $(N+2K')/N^2$ where *K'* is the number of off-diagonal non-zero entries (the aggregate of multiple transmission elements between adjacent buses) and *N* is the number of buses. For example, in a topology with 1000 buses and 1500 transmission elements, *G* and *B* would have a density of $(1000+3000)/1000^2 = .004$. The lowest density for a connected network is the spanning tree. It has $N\text{-}1$ transmission elements and the density is $(N+2(N\text{-}1))/N^2$. For large sparse systems, $(N+2(N\text{-}1))/N^2 \approx 3/N$.

Transformers and FACTS devices change the structure of the Y matrix. If there are transformers and FACTS devices, let

$$\underline{y}_{nmk} = \begin{cases} y_{nmk} & \text{if no transformer} \\ a_{nmk}^2 y_{nmk} & \text{if an ideal transformer} \\ t_{nmk}^* y_{nmk}, \text{ or } -t_{nmk} y_{nmk} v_n & \text{if a phase shifting transformer} \end{cases}$$

as appropriate off-diagonal element, then $y_{nn} = y_{n0} + \sum_{k,m} \underline{y}_{nmk}$, $y_{nm} = \sum_k \underline{y}_{nmk}$, and Y is the matrix $[y_{nm}]$. If there are only ideal in-phase transformers, the Y matrix is symmetric. If there are phase shifting transformers, the symmetry of the Y matrix is lost.

AC Power Flow Equations

Kirchhoff's Current Law. Kirchhoff's current law requires that the sum of the currents injected and withdrawn at bus n equal zero:

$$i_n = \sum_k i_{nmk} \qquad (1)$$

If we define current to ground to be $y_{n0}(v_n - v_0)$ and $v_0 = 0$, we have:

$$i_n = \sum_k y_{nmk}(v_n - v_m) + y_{n0} v_n \qquad (2)$$

$$i_{nmk} = y_{nmk}(v_n - v_m) = g_{nmk}(v^r_n - v^r_m) - b_{nmk}(v^j_n - v^j_m) + \boldsymbol{j}(b_{nmk}(v^r_n - v^r_m) + g_{nmk}(v^j_n - v^j_m))$$

$$i^r_{nmk} = g_{nmk}(v^r_n - v^r_m) - b_{nmk}(v^j_n - v^j_m)$$

$$i^j_{nmk} = b_{nmk}(v^r_n - v^r_m) + g_{nmk}(v^j_n - v^j_m)$$

Current is a linear function of voltage. Rearranging,

$$i_n = v_n(y_{n0} + \sum_k y_{nmk}) - \sum_k y_{nmk} v_m \qquad (3)$$

In matrix notation, the IV flow equations in terms of current (I) and voltage (V) in (3) are

$$I = YV = (G + \boldsymbol{j}B)(V^r + \boldsymbol{j}V^j) = GV^r - BV^j + \boldsymbol{j}(BV^r + GV^j) \qquad (4)$$

where $I^r = GV^r - BV^j$ and $I^j = BV^r + GV^j$

In another matrix format, (4) is

$$I = (I^r, I^j) = \underline{Y}(V^r, V^j)^T \text{ or}$$

$$I = (I^r, I^j) = \begin{bmatrix} G & -B \\ B & G \end{bmatrix} \begin{bmatrix} V^r \\ V^j \end{bmatrix} \quad \text{where } \underline{Y} = \begin{bmatrix} G & -B \\ B & G \end{bmatrix}$$

If a and φ are constant, the $I = YV$ equations are linear. If not, the linearity is lost since some elements of the Y matrix will be functions of V.

Power Flow Equations. The traditional *power-voltage* power flow equations defined in terms of real power (P), reactive power (Q) and voltage (V) are

$$S = P + \boldsymbol{j}Q = diag(V)I^* = diag(V)[YV]^* = diag(V)Y^*V^* \quad (5)$$

The power injections are

$$S = V \bullet I^* = (V^r + \boldsymbol{j}V^j) \bullet (I^r - \boldsymbol{j}I^j) = (V^r \bullet I^r + V^j \bullet I^j) + \boldsymbol{j}(V^j \bullet I^r - V^r \bullet I^j) \quad (6)$$

where

$$P = V^r \bullet I^r + V^j \bullet I^j \quad (7)$$

$$Q = V^j \bullet I^r - V^r \bullet I^j \quad (8)$$

The *power-voltage* power flow equations (5) and (6) are quadratic. The IV flow equations (4) are linear.

Constraints. First, we introduce the physical constraints of generators, load, and transmission.

Generator and Load Constraints. The lower and upper bound constraints for generation (injection) and load (withdrawal) are:

$$P^{min} \leq P \leq P^{max} \quad (9)$$

$$Q^{min} \leq Q \leq Q^{max} \quad (10)$$

In terms of V and I, the injection constraints are:

$$V^r \bullet I^r + V^j \bullet I^j \leq P^{max} \quad (11)$$

$$P^{min} \leq V^r \bullet I^r + V^j \bullet I^j \quad (12)$$

$$V^j \bullet I^r - V^r \bullet I^j \leq Q^{max} \quad (13)$$

$$Q^{min} \leq V^j \bullet I^r - V^r \bullet I^j \quad (14)$$

Inequalities (11)-(14) along with other thermal constraints on equipment enforced at each generator bus constitute a four-dimensional reactive capability curve, also known as a “D-curve’ since it is shaped like the capital letter D, in the PQ space. Additional D-curves defining the tradeoff between real and reactive power constitute a convex set and can be easily linearized (FERC 2005). Equations (11)-(14) are nonconvex quadratic constraints. Since here we model a single period, ramp rates are unnecessary.

Voltage Magnitude Constraints. The two constraints that limit the voltage magnitude in rectangular coordinates at each bus m are

$$(v^r_m)^2 + (v^j_m)^2 \le (v^{max}_m)^2 \quad (15)$$

$$(v^{min}_m)^2 \le (v^r_m)^2 + (v^j_m)^2 \quad (16)$$

Again, each nonlinear inequality involves only the voltage magnitudes at bus m. In matrix terms, the voltage magnitude constraints are:

$$V^r \bullet V^r + V^j \bullet V^j \le (V^{max})^2 \quad (17)$$

$$(V^{min})^2 \le V^r \bullet V^r + V^j \bullet V^j \quad (18)$$

V^{min} and V^{max} are determined by system studies. The voltage magnitude bounds are generally in the range, [.95, 1.05] per unit. High voltages are often constrained by the capabilities of the circuit breakers. Low voltage magnitude constraints can be due to operating requirements of motors or generators.

Line Flow Thermal Constraints. S^{max}_k is a thermal transmission limit on k based on the temperature sensitivity of the conductor and supporting material in the transmission line and transmission elements. Transmission assets generally have three thermal ratings: steady-state, 4-hour and 30-minute. These ratings vary with ambient weather. The apparent power at bus n on transmission element k to bus m is:

$$s_{nmk} = v_n i_{nmk}^* = v_n y^*_{nmk}(v_n - v_m)^*. = v_n y^*_{nmk} v^*_n - v_n y^*_{nmk} v^*_m)$$

The thermal limit on s_{nmk} is

$$(s^r_{nmk})^2 + (s^j_{nmk})^2 = |s_{nmk}|^2 \le (s^{max}_k)^2 \quad (19)$$

These constraints are quadratic in s^r_{nmk} and s^j_{nmk} and quartic in v^r_n, v^j_n, v^r_m, v^j_m. Since

$$v_n = v^r_n + \boldsymbol{j} v^j_n \text{ and } y_{nmk} = g_{nmk} + \boldsymbol{j} b_{nmk},$$

$$v_n y^*_{nmk} v^*_n = (v^r_n + \boldsymbol{j} v^j_n)(g_{nmk} + \boldsymbol{j} b_{nmk})(v^r_n + \boldsymbol{j} v^j_n)$$

Expanding, we obtain

$$v_n y^*_{nmk} v^*_n = [g_{nmk} v^r_n - b_{nmk} v^j_n + \boldsymbol{j}(g_{nmk} v^j_n + b_{nmk} v^r_n)](v^r_n + \boldsymbol{j} v^j_n)$$

Expanding again, we obtain

$$v_n y^*_{nmk} v^*_n = g_{nmk}(v^r_n v^r_n - v^j_n v^j_n) - b_{nmk}(v^r_n v^j_n + v^r_n v^j_n)$$
$$+\boldsymbol{j}[g_{nmk}(v^j_n v^r_n + v^j_n v^r_n) + b_{nmk}(v^r_n v^r_n - v^j_n v^j_n)]$$

$$v_n y^*_{nmk} v^*_n = g_{nmk}(v^r_n v^r_n - v^j_n v^j_n) - 2b_{nmk}(v^r_n v^j_n)$$
$$+\boldsymbol{j}[2g_{nmk}(v^j_n v^r_n) + b_{nmk}(v^r_n v^r_n - v^j_n v^j_n)] \quad (20)$$

In matrix notation,

$$Re(v_n y^*_{nmk} v^*_n) = [v^r_n, v^j_n] \begin{bmatrix} g_{nmk} & -b_{nmk} \\ -b_{nmk} & -g_{nmk} \end{bmatrix} \begin{bmatrix} v^r_n \\ v^j_n \end{bmatrix}$$

$$Im(v_n y^*_{nmk} v^*_n) = [v^r_n, v^j_n] \begin{bmatrix} b_{nmk} & g_{nmk} \\ g_{nmk} & -b_{nmk} \end{bmatrix} \begin{bmatrix} v^r_n \\ v^j_n \end{bmatrix}$$

Similarly, $v_n y^*_{nmk} v^*_m = (v^r_n + \boldsymbol{j} v^j_n)(g_{nmk} + \boldsymbol{j} b_{nmk})(v^r_m + \boldsymbol{j} v^j_m)$

Expanding, we obtain

$$= [g_{nmk} v^r_n - b_{nmk} v^j_n + \boldsymbol{j}(g_{nmk} v^j_n + b_{nmk} v^r_n)](v^r_m + \boldsymbol{j} v^j_m)$$

Expanding and collecting terms,

$$= g_{nmk}(v^r_n v^r_m + v^j_n v^j_m) + b_{nmk}(v^j_n v^r_m - v^r_n v^r_m) + \boldsymbol{j}[g_{nmk}(v^j_n v^r_m - v^j_n v^j_m) + b_{nmk}(v^r_n v^r_m - v^j_n v^j_m)]$$

(21)

In matrix notation,

$$Re(v_n y^*_{nmk} v^*_m) = [v^r_m, v^j_m] \begin{bmatrix} g_{nmk} & -b_{nmk} \\ b_{nmk} & g_{nmk} \end{bmatrix} \begin{bmatrix} v^r_n \\ v^j_n \end{bmatrix}$$

$$Im(v_n y^*_{nmk} v^*_m) = [v^r_m, v^j_m] \begin{bmatrix} b_{nmk} & -g_{nmk} \\ g_{nmk} & -b_{nmk} \end{bmatrix} \begin{bmatrix} v^r_n \\ v^j_n \end{bmatrix}$$

Inequality (19) becomes a quadratic constraint.

Line Flow Constraints as Current Limitations. As current increases, lines sag and equipment may be damaged by overheating. The constraints that limit the current magnitude in rectangular coordinates at each bus n on k are

$$(i^r_{nmk})^2 + (i^j_{nmk})^2 \leq (i^{max}_{nmk})^2 \qquad (23)$$

Again, the nonlinearities are convex quadratic and isolated to the complex current at the bus. Generally, the maximum currents, i^{max}_{nmk}, are determined by material science properties of conductors and transmission equipment, or as a result of system stability studies.

Line Flow Constraints as Voltage Angle Constraints. The power flowing over an AC line is approximately proportional to the sine of the voltage phase angle difference at the receiving and transmitting ends. For stability reasons, the voltage angle difference for terminal buses n and m connected by transmission element k can be constrained as follows:

$$\theta^{min}_{nm} \leq \theta_n - \theta_m \leq \theta^{max}_{nm} \qquad (24)$$

In the rectangular formulation, the arctan function appears in some constraints.

$$\theta^{min}_{nm} \leq arctan(v^j_n/v^r_n) - arctan(v^j_m/v^r_m) \leq \theta^{max}_{nm} \qquad (25)$$

The theoretical steady-state stability limit for power transfer between two buses across a lossless line is 90 degrees. If this limit were exceeded, synchronous machines at one end of the line would lose synchronism with the other end of the line. In addition, transient stability and relay limits on reclosing lines constrain voltage angle differences. The angle constraints used in the ACOPF should be the smallest of these angle constraints, which depend on the equipment installed and configuration. However, many test cases do not include any voltage angle or line flow constraints. In general, system engineers design and operate the system comfortably below the voltage angle limit to allow time to respond if the voltage angle difference across any line approaches its limit.

5. ACOPF Formulations

We begin with a discussion of objective functions, then a note on bus types, and finally discuss different formulations of the ACOPF. The formulations of the ACOPF presented here include all the constraints, but may take different approaches to modeling the constraints. As discussed above, current, voltage magnitude, and voltage angle constraints can be calculated that are surrogates for each other. We discuss constraints further in (O'Neill 2012).

Objective Function. Various authors formulate the ACOPF with different objective functions. They include minimizing generation costs, maximizing market surplus, minimizing losses, minimizing generation (equivalent to minimizing losses), and maximizing transfers. Without demand functions, minimizing generation costs and maximizing market surplus are equivalent.

A full ACOPF that accurately models all constraints and controls with an objective function of minimizing cost would inherently meet the objectives of minimizing generator fuel costs, minimizing generation output, minimizing losses, minimizing load shedding, and minimizing control actions.

When it is not feasible to run a full ACOPF due to time constraints, computing power, or lack of a robust solution algorithm, a common substitute is to decouple the problem and iterate between a DCOPF that minimizes costs by varying real power, then fix the generator outputs from the DCOPF and run an ACOPF that minimizes losses by varying reactive power of generators, capacitors, etc. For economically dispatching resources in an ACOPF that fully models voltage and stability constraints, minimizing cost is the correct objective function; objective functions of minimizing losses, minimizing generation, and maximizing transfers for an ACOPF are inconsistent with economic principles, and result in sub-optimal dispatch. We do not discuss the details of decoupled OPF here, but save it for a future review of solution algorithms.

Stott et al. discuss badly-posed problems when an OPF formulation does not adhere to the normal engineering principles of power system operation (Stott 1987). They mention a few examples in decoupled formulations: minimizing losses with generator real power output as variables would move away from a minimum-cost solution; imposing limits on MW reserves with only generator voltage controls and transformer voltage tap controls, but no real power control to meet the reserve limit. They state that it is helpful to associate each control, constraint, and objective in a decoupled OPF with either or both the active and reactive power subproblems. They further note that some objective functions and constraints are not algebraic or differentiable, and that multiple solutions are likely to exist, in particular when there are many reactive power controls (such as switched capacitors, FACTS devices, or generators) in network loops.

It is possible to formulate an objective function that includes the cost of reactive power. For a generator the cost of generation is a function of the apparent power generated, $c(S) = c_P(P) + c_Q(Q)$, where $S = (P^2 + Q^2)^{1/2}$. If we assume that the cost of reactive power is small compared to the cost of real power and if the cost function, $c(S)$, is linear in S, an approximation of $c(S)$ is

$$c(S) \approx c_P(P) + c_Q(|Q|).$$

Bus-type. In P, Q, |V|, θ space, there are four quantities at each bus: voltage magnitude (V), voltage angle (θ), real power (P), and reactive power (Q). In a power flow solution without optimization, buses are classified into three bus types: PQ, PV and slack. PQ buses generally correspond to loads and PV buses to generators.

Generator buses are called PV buses because power and voltage magnitude are fixed; load buses are known as PQ buses because real and reactive power are fixed, that is, $P^{min} = P^{max}$ and $Q^{min} = Q^{max}$; slack or reference buses have a fixed voltage magnitude and voltage angle. For a power flow to solve, the slack bus needs to have sufficient real and reactive power to make up for system losses and hold the slack bus voltage magnitude at 1; for this reason, a bus with a large generator is commonly chosen as a slack bus. Table 3 compares the different types of buses.

Table 3: Bus classification used in power flow problems

Bus Type	**Fixed quantities**	**Variable quantities**	**Physical interpretation**
PV	real power, voltage magnitude	reactive power, voltage angle	generator
PQ	real power, reactive power	voltage magnitude, voltage angle	load, or generator with fixed output
Slack	voltage magnitude, voltage angle	real power, reactive power	an arbitrarily chosen generator

In a power flow, the slack bus serves partly to ensure an equal number of variables and constraints; without a designated slack bus, the system would be over-determined, with more equations than unknowns. Stott states that the need for a slack bus also arises because the system I^2R losses are not precisely known in advance of the load-flow calculation for linear DC models and therefore cannot be assigned to a particular generator dispatch (Stott 1974). Some models use a distributed slack bus where generators at several different buses provide system slack.

We note that an ACOPF that iterates between a simplified OPF and an AC power flow may need a slack bus for the power flow iterations, but even then the voltage magnitude at the slack bus does not have to be fixed.

When using an iterative method such as Newton or Gauss-Seidel to solve the power flow equations, the convergence tolerance is generally set based on the "mismatch" terms. Mismatch refers to the difference between known values at each bus, such as P and Q at load buses, and the values $P(x)$ and $Q(x)$ computed with the power flow equations at each iteration.

Since the ACOPF is an optimization problem, where the number of variables does not have to equal the number of constraints, specifying a slack or reference bus is unnecessary. In fact, Carpentier noted this as early as 1962 (Carpentier 1962).[1] In all optimization formulations herein, we forgo the bus type designation. In an optimization context, these categorizations seem overly prescriptive, and could unnecessarily over-constrain the problem. For example, fixing the reference voltage magnitude at 1.0 per unit when in normal operations generators vary voltage magnitude between 0.95 and 1.05 per unit could result in a sub-optimal solution. Most modern solvers pre-process the problem, removing variables that have equal lower and upper bounds and replacing them with a constant.

ACOPF Power-Voltage (PQV) Formulation. Most of the ACOPF literature uses the polar *power-voltage* formulations based on the early work of Carpentier during the 1960's (Carpentier 1962).

Polar Power-Voltage Formulation. The polar *power-voltage* (polar PQV) ACOPF (polar ACOPF-PQV) replaces quadratic equality constraints in (32) with the polar formulation of (27)-(28):

Network-wide objective function: $Min\ c(S)$ (26)

Network-wide constraints:

$$P_n = \sum_{mk} V_n V_m (G_{nmk} cos\theta_{nm} + B_{nmk} sin\theta_{nm}) \quad (27)$$

$$Q_n = \sum_{mk} V_n V_m (G_{nmk} sin\theta_{nm} - B_{nmk} cos\theta_{nm}) \quad (28)$$

$$V^{min} \le V \le V^{max} \quad (29)$$

1 Rough translation of (Carpentier 1962): If voltage and angle are taken as variables in place of P and Q, the restriction of fixing the reference voltage can be lifted; voltage and angle are in effect independent variables that fix the state of the network, and it suffices to write an objective function that is minimized with respect to these variables. The arbitrarily chosen reference bus disappears and the problem is the most general that one can pose.

$$\theta^{min}_{nm} \leq \theta_n - \theta_m \leq \theta^{max}_{nm}. \tag{30}$$

In this formulation, (27) and (28) represent 2N nonlinear equality constraints with quadratic terms and sine and cosine functions that apply throughout the network. In this formulation, voltage angle difference constraints are linear. In the rectangular formulation discussed below, arctan functions appear in the angle difference constraints.

Rectangular Power Voltage Formulation. The rectangular *power-voltage* formulation, shown below, is less common in the literature. The rectangular *power-voltage* (rectangular PQV) ACOPF (rectangular ACOPF-PQV) formulation is shown below.

Network-wide objective function: $Min\ c(S)$ (31)

Network-wide constraint: $P + \boldsymbol{j}Q = S = V \bullet I^* = V \bullet Y^* V^*$ (32)

Bus-specific constraints

$$P^{min} \leq P \leq P^{max} \tag{33}$$

$$Q^{min} \leq Q \leq Q^{max} \tag{34}$$

$$(|s_{nmk}|)^2 \leq (s^{max}_k)^2 \qquad \text{for all } k \tag{35}$$

(29) is replaced by:

$$V^r \bullet V^r + V^j \bullet V^j \leq (V^{max})^2 \tag{36}$$

$$(V^{min})^2 \leq V^r \bullet V^r + V^j \bullet V^j \tag{37}$$

(30) is replaced by:

$$\theta^{min}_{nm} \leq arctan(v^j_n / v^r_n) - arctan(v^j_m / v^r_m) \leq \theta^{max}_{nm} \tag{38}$$

In this formulation, (32) represents *2N* quadratic equalities that apply throughout the network; (33)-(34) are simple variable bounds at each bus; (35) and (37) represents convex quadratic inequalities at each bus; (37) represents a nonconvex quadratic inequalitiy at each bus; and (37) and (38) represents nonconvex inequalities between each set of connected buses.

ACOPF Current Injection (IV) Formulation. Current injection formulations use power flow equations based on current and voltage rather than power flow equations based on power and voltage discussed above. We only consider the rectangular current-voltage (rectangular IV) ACOPF (rectangular ACOPF-IV) formulation due to the advantages in expressing the current injections as linear equality constraints; however, the polar current-voltage formulation could be easily derived.

The IV formulation has *6N* variables (P, Q, V^r, V^j, I^r, I^j) and the VΘ has *4N* variables ($P, Q, |V|, \Theta$).

Rectangular ACOPF-IV formulation. The rectangular ACOPF-IV formulation is shown below.

Network-wide objective function: *Min c*(*S*) (40)

Network-wide constraint: $I = YV$ (41)

Bus-specific constraints:

$$P = V^r \bullet I^r + V^j \bullet I^j \leq P^{max} \quad (42)$$

$$P^{min} \leq P = V^r \bullet I^r + V^j \bullet I^j \quad (43)$$

$$Q = V^j \bullet I^r - V^r \bullet I^j \leq Q^{max} \quad (44)$$

$$Q^{min} \leq Q = V^j \bullet I^r - V^r \bullet I^j \quad (45)$$

$$V^r \bullet V^r + V^j \bullet V^j \leq (V^{max})^2 \quad (46)$$

$$(V^{min})^2 \leq V^r \bullet V^r + V^j \bullet V^j \quad (47)$$

$$(i_{nmk})^2 \leq (i^{max}{}_k)^2 \qquad \text{for all } k \quad (48)$$

$$\theta^{min}{}_{nm} \leq arctan(v^j{}_n/v^r{}_n) - arctan(v^j{}_m/v^r{}_m) \leq \theta^{max}{}_{nm} \quad (49)$$

In this formulation, (41) represents *2N* linear equality constraints that apply throughout the network. This is in contrast to the PQV formulations where quadratic and trigonometric constraints apply throughout the network and linear constraints are isolated at each bus. Equations (42) to (45) are local quadratic nonconvex constraints. Equations (46) and (48) are local convex quadratic inequality constraints, but (47) are non-convex local quadratic inequality constraints. Overall, the constraint set is still nonconvex, but we hypothesize that this formulation may be easier to solve than the power-voltage formulations, since the nonlinearities are isolated to each bus and each transmission element, while the constraints that apply throughout the network are linear. In general, linear solvers solve problems faster than nonlinear solvers. As discussed previously, the voltage angle limit (49) could be replaced with an analogous current limit and the problem becomes locally quadratic with linear network equations, and (48) and (49) are essentially redundant constraints.

	Polar PQV	**Rectangular PQV**	**Rectangular IV**
Network constraints	2N nonlinear equality constraints with quadratic terms and sine and cosine functions	2N quadratic equalities	2N **linear** equality constraints
Voltage angle difference constraints	**Linear**	Nonconvex (arctan)	**Linear** if replaced with current or apparent power constraint
Bus constraints	**Linear**	Noncovex quadratic inequalities	Locally quadratic, some nonconvex, some convex

6. Literature Review of Formulations

Most literature uses the polar *power-voltage* formulation, while a smaller group of papers use the rectangular *power-voltage* formulation. Some have also proposed hybrid and alternative formulations. So, rather than attempt to review the vast literature on the traditional formulation based on power and reactive power equations, we focus on alternative formulations in this section.

Stott et al. criticize that much OPF research since the classical formulations of Carpentier, Dommel and Tinney have addressed similar formulations without considering the additional requirements needed for practical real-time applications, partly because OPF problems are still stretching the limits of applied optimization technology, and also that utilities have been slow to adopt software to calculate OPF "on-line," or in near-real-time (Stott 1987). They further note that it is a mistake to analytically formulate OPF problems by regarding them as simple extensions of conventional power flow; once the power flow problem is formulated as an optimization problem with degrees of freedom, problems that appear easy to solve can turn out to be badly posed, for example with conflicting objective function, controls, and constraints. For OPF, they note that researchers have not agreed on "rules of solvability," which are the engineering criteria needed for an OPF solution to be physically valid, especially for voltage and reactive power, and that these "rules of solvability" have hardly if ever been mentioned in the vast literature on OPF. They also identify several common problems with the OPF formulation. Most of these relate to modeling voltage characteristics of generation, load, and transformers, but also include problems with incompatibility of objective, controls, and constraints. For example, one incompatibility problem uses an objective of minimizing losses with generator real power outputs as variables, rather than fixing generator real power outputs at the minimum cost dispatch and adjusting reactive power settings to minimize losses (Stott 1987).

A few researchers have developed a current injection formulation for the power flow or optimal power flow equations. Current injection and reactive current are terms used in the literature for a formulation similar to the IV formulation discussed earlier in this paper. Additionally, some literature uses the term "in phase" for the real component of current (I^r) and "quadrature" for the imaginary component of current (I^j); in this context, quadrature refers to being 90 degrees out of phase. Most of these papers identify challenges modeling generator, or PV buses, where the real power injection and voltage magnitude are known but the reactive power injection is not. Several authors have identified ways to model PV buses. We discuss these formulations here.

Dommel et al. present a power flow formulation using current injections and a mix of polar and rectangular coordinates, where each PQ bus is represented by two equations for the real and imaginary components of current mismatches in terms of complex voltage in rectangular coordinates, while PV buses are represented by a single active power mismatch equation and associated voltage angle deviation (Dommel 1970). Tinney later mentions that a current injection algorithm with a constant nodal admittance matrix could not be used for general power flow applications because a satisfactory method of modeling PV buses had not been developed (Tinney 1991). Other authors allude to difficulties modeling PV buses using current injections, and much of literature using current injection formulations applies to radial distribution networks where PV buses are less common. For some solution techniques, modeling PV buses with current injection equations introduces singularities into some matrices in the solution technique. Substitutions introduce dependencies in the Jacobian, meaning that the entire Jacobian would have to be recalculated at each step (Gómez Romero 2002). Various authors have proposed substitutions and approximations to model PV buses in a current injection formulation.

Stadlin and Fletcher discuss a "voltage versus reactive current" model for voltage and reactive control that is well suited for use with a linear programming algorithm (Stadlin 1982). This paper does not directly discuss an OPF, but provides a model that could be used in a linear programming optimization for reactive dispatch and voltage control. The model would be used after a real power dispatch model, such as a decoupled power flow, was run, and would assume fixed real power generation, except at the swing bus. This model uses real and reactive current (computed as P/V and Q/V, respectively). The authors use an incremental current model rather than an incremental power model because the Jacobian matrices of a current model are less sensitive to bus voltage variations. In addition, the sensitivity coefficient of voltage to reactive current is much less sensitive than the sensitivity coefficient of voltage to reactive power. The authors fix the swing bus voltage angle at zero, but allow the voltage magnitude to float. The authors note their assumptions result in a more accurate "decoupled" relationship between incremental reactive current and voltage than is given by the B matrix used in B-θ decoupled OPF, and that this more accurate and more linear model reduces the iterations in an optimization algorithm. The sensitivity coefficients in the B matrix are accurate only in a small range of voltage, requiring recalculation of the B matrix for large changes in voltage; Stadlin and Fletcher's model is accurate and linear over a larger voltage operating range than a B-θ model. Stadlin and Fletcher wanted to define a model which remains nearly linear for changes in voltage and reactive variables so that efficient linear programming techniques could be applied.

Da Costa and Rosa note that for the rectangular formulation, generation or PV buses have different equations than load or PQ buses. At load buses, active and reactive power mismatches are known. At generation buses, reactive power mismatches are not known but voltage magnitude constraints are known, because in a traditionally formulated power flow, generator reactive power output is variable (Da Costa 2008). Therefore, a voltage magnitude constraint is added to each load bus, resulting in a different Jacobian matrix.

Da Costa et al. present a rectangular formulation of a Newton-Raphson power flow based on current injections, for both PQ and PV buses (Da Costa 1999, Lin 2008). In this formulation, the Jacobian matrix has the same structure as the nodal admittance matrix, except for PV buses. For PV buses, the authors introduce a new dependent variable, ΔQ, and an additional constraint on voltage magnitude deviation. The voltage magnitude constraint is linearized:

$\Delta V_n = 0 \approx (V^r{}_n/V_n)\Delta V^r{}_n + (V^j{}_n/V_n)\Delta V^j{}_n$, where V_n is the voltage magnitude at bus n, $V^r{}_n$ is the real component of voltage at bus n, and $V^j{}_n$ is the imaginary component of voltage at bus n.

Da Costa and Rosa note that the current injection equations are linear for electrical networks with only PQ buses and a constant impedance load model (Da Costa 2008).

Jiang et al. published a power-current hybrid rectangular OPF formulation. They divide buses into two types, those with non-zero injections, and those with zero injections (Jiang 2009). For buses with non-zero injections, the power mismatch formulation is used, while the current mismatch formulation is used for buses with zero injections. The authors note that in the current mismatch formulation, which is similar to the IV formulation presented above, the first-order derivatives of the equations are constants and the second-order derivatives are zeros. By dividing the buses into two groups, the hybrid method saves computation time for the Jacobian and Hessian matrices.

Meliopoulos and Tao use a formulation referred to as "Quadratic Power Flow," with current conservation equations from Kirchhoff's current law in rectangular coordinates instead of power flow equations, and add operational constraints to the model only when they are violated in the previous iteration (Meliopoulos 2011). The equations modeling generators, constant power loads, and transformers are quadratic equations separated into real and imaginary parts. The objective function is to minimize the sum of a penalty factor times the sum of current mismatches and the total generator costs. The model includes a slack bus as the "mismatch current source" where the voltage magnitude is a state variable and the real and imaginary components of complex voltage are control variables, while a PV bus has the voltage magnitude as a control variable and real and imaginary components of complex voltage as state variables. The authors linearize to eliminate integer state variables. The quadratic constraints are linearized when they are added to the model.

7. Conclusions

This paper has presented a literature review of different formulations of the ACOPF and discussed areas for future research where the ACOPF could be improved. The ACOPF problem is inherently difficult due to nonconvexities, multipart nonlinear pricing, and alternating current. We do not yet have practical approaches to solving nonconvex problems. The ACOPF is a well-structured problem, and has developed during 50 years of research. Academia and industry have developed various approaches to solving the ACOPF, with different formulations, algorithms, and assumptions. The traditional approach has been to linearize the full ACOPF problem and decompose it into subproblems. The ACOPF is not a hypothetical problem – it is solved every 5 minutes through approximations and judgment. After 50 years, there is not yet a commercially viable full ACOPF. Many possibilities and ways to examine the ACOPF remain. Today's solvers do not return the gap between the given and globally optimal solution; if we make a rough estimate that today's solvers are on average off by 10%, and world energy costs are $400 billion, closing the gap by 10% is a huge financial impact.

References

J. Carpentier, “Contribution á l’étude du dispatching économique,” Bulletin de la Société Française des Électriciens, ser. 8, vol. 3, pp. 431-447, 1962.

J. Carpentier, “Optimal power flows,” International Journal of Electrical Power and Energy Systems, Vol. 1, Issue 1, pp. 3-15, Apr. 1979.

V.M. da Costa, N. Martins, J.L.R. Pereira, “Developments in the Newton Raphson Power Flow Formulation Based on Current Injections,” IEEE Transactions on Power Systems, Vol. 14, No. 4, pp. 1320-1326, Nov. 1999.

V.M. da Costa and A.L.S. Rosa, “A Comparative Analysis of Different Power Flow Methodologies,” IEEE Transmission & Distribution Conference and Exposition: Latin America,” Bogota, Aug. 13-15, 2008.

H.W. Dommel, W.F. Tinney, and W.L. Powell, “Further Developments in Newton’s method for power system applications,” IEEE Winter Power Meeting, Conference Paper No. 70 CP 161-PWR, Jan. 1970, cited in V.M. da Costa, N. Martins, J.L.R. Pereira, “Developments in the Newton Raphson Power Flow Formulation Based on Current Injections,” IEEE Transactions on Power Systems, Vol. 14, No. 4, p. 1320, Nov. 1999.

U.S. Department of Energy, Energy Information Administration (EIA), International Energy Statistics, 2012, available at http://www.eia.gov/cfapps/ipdbproject/IEDIndex3.cfm?tid=2&pid=2&aid=2; also Wholesale Market Data, 2012, available at http://www.eia.gov/electricity/wholesale/.

FERC Staff, “Principles for Efficient and Reliable Reactive Power Supply and Consumption,” available at http://www.ferc.gov/eventcalendar/files/20050310144430-02-04-05-reactive-power.pdf, Feb. 2005.

FERC Joint Boards on Security Constrained Economic Dispatch, “Study and Recommendations Regarding Security Constrained Economic Dispatch (SCED) in the Northeast by the Joint Board on Economic Dispatch for the Northeast Region,” FERC Docket AD05-13, page 15, available at http://www.ferc.gov/industries/electric/indus-act/joint-boards/northeast-recom.pdf, 2006.

FERC Staff, "Recent ISO Software Enhancements and Future Modeling Plans," available at http://www.ferc.gov/industries/electric/indus-act/rto/rto-iso-soft-2011.pdf, Nov. 2011.

FERC and NERC staff, "Arizona-Southern California Outages on September 8, 2011: Causes and Recommendations," pp. 23-59, Apr. 2012.

FERC Order, Midwest Independent System Operator, 138 FERC ¶ 61,235, FERC Docket ER12-678 (2012).

A. Gómez Expósito, E. Romero Ramos, "Augmented Rectangular Load Flow Model," IEEE Transactions on Power Systems, Vol. 17, No. 2, pp. 271-274, May 2002.

H.H. Happ, "Optimal Power Dispatch – A Comprehensive Survey," IEEE Transactions on Power Apparatus and Systems, Vol. PAS-96, no. 3, pp. 841-844, May/Jun. 1977.

I.A. Hiskens and R.J. Davy, "Exploring the Power Flow Solution Space Boundary," IEEE Transactions on Power Systems, Vol. 16, No. 3, pp. 389-395, Aug. 2001.

M. Huneault and F.D. Galiana, "A Survey of the Optimal Power Flow Literature," IEEE Transactions on Power Systems, Vol. 6 No. 2, May 1991, pp. 762-770.

Q.Y. Jiang, H.D. Chiang, C.X. Guo, Y.J. Cao, "Power-current hybrid rectangular formulation for interior-point optimal power flow," Institute of Engineering and Technology (IET) Generation, Transmission and Distribution, Vol. 3, Iss. 8, pp. 748-756, 2009.

L.K. Kirchmayer, *Economic Operation of Power Systems*, Wiley and Sons 1958.

W.M. Lin, C.H. Huang, T.S. Zhan, "A Hybrid Current-Power Optimal Power Flow Technique," IEEE Transactions on Power Systems, Vol. 23, No. 1, pp. 177-185, Feb. 2008.

J.A. Momoh, R.J. Koessler, M.S. Bond, B. Stott, D. Sun, A. Papalexopoulos, P. Ristanovic, "Challenges to Optimal Power Flow," IEEE Transactions on Power Systems, Vol. 12, No. 1, pp. 444-447, Feb. 1997.

R.P. O'Neill, "The IV Formulation of the ACOPF and its Linearizations," FERC Technical conference to discuss opportunities for increasing real-time and day-ahead market efficiency through improved software , Docket No. AD10-12-003, available at http://www.ferc.gov/EventCalendar/Files/20120626084555-Tuesday_SessionTB_ONeill.ppt.pdf, June 26, 2012.

R. P. O'Neill, T. Dautel and E. Krall, Recent ISO Software Enhancements and Future Software and Modeling Plans, Staff Report, Federal Energy Regulatory Commission, available at http://www.ferc.gov/industries/electric/indus-act/rto/rto-iso-soft-2011.pdf, November 2011, and Private communication with ISO personnel.

J. Peschon, D.S. Piercy, W.F. Tinney, O.J. Tveit, M. Cuénod, "Optimum Control of Reactive Power Flow," IEEE Transactions on Power Apparatus and Systems, Vol. PAS-87, No. 1, pp. 40-48, Jan. 1968.

PJM, "PJM's Striving for Perfect Dispatch Nets Nearly $200 Million in Savings," News Release, Jan. 19, 2012, available at https://www.pjm.com/~/media/about-pjm/newsroom/2012-releases/20120119-perfect-dispatch-news-release-2012.ashx.

J. Riquelme Santos, A. Gómez Expósito, J.L. Martínez Ramos. "Slack Bus Selection to Minimize the System Power Imbalance in Load-Flow Studies." IEEE Transactions on Power Systems, 19(2), May 2004.

A.M. Sasson and F.J. Jaimes, "Digital Methods Applied to Power Flow Studies," IEEE Trans. on Power Apparatus and Systems, vol. 86, No. 7, pp. 860-867, Jul. 1967.

A. Schecter, "Exploration of the ACOPF Feasible Region for the Standard IEEE Test Set," FERC Technical conference to discuss opportunities for increasing real-time and day-ahead market efficiency through improved software , Docket No. AD10-12-003, available at http://www.ferc.gov/EventCalendar/Files/20120626084625-Tuesday,%20Session%20TC-1,%20Schecter%20.pdf, June 26, 2012.

R.B. Squires, "Economic Dispatch of Generation Directly from Power System Voltages and Admittances," AIEE Trans. vol. 79, pt. III, pp. 1235-1244, 1961.

W.O. Stadlin and D.L. Fletcher, "Voltage Versus Reactive Current Model for Dispatch and Control," IEEE Transactions on Power Apparatus and Systems, Vol. PAS-101, NO. 10, pp. 3751-3760, Oct. 1982.

B. Stott, "Review of Load-Flow Calculation Methods," Proceedings of the IEEE, vol. 62 no. 7, page 916, Jul. 1974.

B. Stott, O. Alsac, and A.J. Monticelli, “Security Analysis and Optimization,” Proceedings of the IEEE, vol. 75, no. 12, pp. 1623-1644, 1987.

Y. Tao and A.P.S. Meliopoulous , “Optimal Power Flow via Quadratic Power Flow,” IEEE Power Systems Conference and Exposition, Phoenix, May 2011.

W.F. Tinney, J.M. Bright, K.D. Demaree, B.A. Hughes, “Some Deficiencies in Optimal Power Flow,” IEEE Transactions on Power Systems, Vol. 3 No. 2, pp. 676-683, May 1988.

W.F. Tinney, “A presentation to the workshop in engineering mathematics and computer sciences,” EPRI publication EAR/EL-7/07, pages 1-15, August 1991, cited in W.-M. Lin, C.-H. Huang, T.-S. Zhan, “A Hubrid Current-Power Optimal Power Flow Technique,” IEEE Transactions on Power Systems, Vol. 23, No. 1, p. 177, Feb. 2008.

J.B. Ward and H.W. Hale, “Digital Computer Solution of Power Flow Problems,” Trans. AIEE (Power Apparatus and Systems), vol. 75, pp. 398-404, Jun. 1956.